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# KEY TO MILNE'S GEOMETRY

## BOOK I

**Ex. 1.** Find the complement of an angle of 15°; 27°; 35°; 49°.

**Solution.** $90^\circ - 15^\circ = 75^\circ$; $90^\circ - 27^\circ = 63^\circ$; $90^\circ - 35^\circ = 55^\circ$; and $90^\circ - 49^\circ = 41^\circ$.

**Ex. 2.** Find the supplement of an angle of 38°; 96°; 114°.

**Solution.** $180^\circ - 38^\circ = 142^\circ$; $180^\circ - 96^\circ = 84^\circ$; and $180^\circ - 114^\circ = 66^\circ$.

**Ex. 3.** The complement of an angle is 63°. What is the angle?

**Solution.** The angle $= 90^\circ - 63^\circ = 27^\circ$.

**Ex. 4.** The supplement of an angle is 103°. What is the angle?

**Solution.** The angle $= 180^\circ - 103^\circ = 77^\circ$.

**Ex. 5.** Find the complement of the supplement of an angle of 165°; 140°; 122°; 113°; 108°; 99°.

**Solution.**

(1) Sup. $= 180^\circ - 165^\circ = 15^\circ$; $\therefore$ comp. of sup. $= 90^\circ - 15^\circ = 75^\circ$.
(2) Sup. $= 180^\circ - 140^\circ = 40^\circ$; $\therefore$ comp. of sup. $= 90^\circ - 40^\circ = 50^\circ$.
(3) Sup. $= 180^\circ - 122^\circ = 58^\circ$; $\therefore$ comp. of sup. $= 90^\circ - 58^\circ = 32^\circ$.
(4) Sup. $= 180^\circ - 113^\circ = 67^\circ$; $\therefore$ comp. of sup. $= 90^\circ - 67^\circ = 23^\circ$.
(5) Sup. $= 180^\circ - 108^\circ = 72^\circ$; $\therefore$ comp. of sup. $= 90^\circ - 72^\circ = 18^\circ$.
(6) Sup. $= 180^\circ - 99^\circ = 81^\circ$; $\therefore$ comp. of sup. $= 90^\circ - 81^\circ = 9^\circ$.

**Ex. 6.** Find the supplement of the complement of an angle of 48°; 84°; 27°; 16°; 31°; 54°; 39°.

**Solution.**

(1) Comp. $= 90^\circ - 48^\circ = 42^\circ$; $\therefore$ sup. of comp. $= 180^\circ - 42^\circ = 138^\circ$.
(2) Comp. $= 90^\circ - 84^\circ = 6^\circ$; $\therefore$ sup. of comp. $= 180^\circ - 6^\circ = 174^\circ$.
(3) Comp. $= 90^\circ - 27^\circ = 63^\circ$; $\therefore$ sup. of comp. $= 180^\circ - 63^\circ = 117^\circ$.
(4) Comp. $= 90^\circ - 16^\circ = 74^\circ$; $\therefore$ sup. of comp. $= 180^\circ - 74^\circ = 106^\circ$.
(5) Comp. $= 90^\circ - 31^\circ = 59^\circ$; $\therefore$ sup. of comp. $= 180^\circ - 59^\circ = 121^\circ$.
(6) Comp. $= 90^\circ - 54^\circ = 36^\circ$; $\therefore$ sup. of comp. $= 180^\circ - 36^\circ = 144^\circ$.
(7) Comp. $= 90^\circ - 39^\circ = 51^\circ$; $\therefore$ sup. of comp. $= 180^\circ - 51^\circ = 129^\circ$.

**Ex. 7.** One line meets another, making two angles with it. One angle contains 87°. How many degrees are there in the other?

**Solution.** § 55, the sum of the $\measuredangle s = 180°$; $\therefore$ the required $\angle = 180° - 87° = 93°$.

**Ex. 8.** Four of the five consecutive angles about a point contain 17°, 36°, 89°, and 110° respectively. How many degrees are there in the fifth angle?

**Solution.** The sum of all the angles about a point $= 360°$; $\therefore$ the required angle $= 360° - (17° + 36° + 89° + 110°) = 108°$.

**Ex. 9.** If two lines meet a third line at the same point, making with the third line angles of 27° and 63° respectively, what is the angle between the two lines?

**Solution.** Let $DB$ and $EB$ make $\measuredangle s$ of 27° and 63° respectively, with the line $AC$ at the point $B$.

Then,

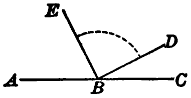
(1) $\angle EBD = 180° - (\angle ABE + CBD)$
$= 180° - (63° + 27°) = 90°$.

(2) $\angle EBD = \angle CBE - \angle CBD = 63° - 27° = 36°$.

(3) $\angle EBD = \angle CBE + \angle CBD = 63° + 27° = 90°$.

(4) $\angle EBD = \angle ABD + \angle ABE$;

but $\angle ABE = 180° - \angle CBE = 180° - 63° = 117°$, and data, $\angle ABD = 27°$; hence, substituting, $\angle EBD = 27° + 117° = 144°$.



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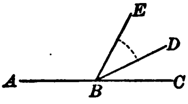

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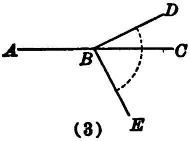

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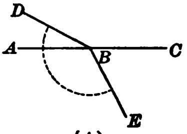

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**Ex. 10.** The complement of an angle is 43°. What is the supplement of the angle?

**Solution.** The angle $= 90° - 43° = 47°$; $\therefore$ sup. of the angle $= 180° - 47° = 133°$.

**Ex. 11.** The supplement of an angle is 125°. What is the complement of the angle?

**Solution.** The angle $= 180° - 125° = 55°$; $\therefore$ comp. of the angle $= 90° - 55° = 35°$.

**Ex. 12.** How many degrees are there in the supplement of the complement of an angle of 60°? Of 43° 25′ 50″?

**Solution.**

(1) Comp. $= 90° - 60° = 30°$; $\therefore$ sup. of comp. $= 180° - 30° = 150°$.

(2) Comp. $= 90° - 43° 25' 50'' = 46° 34' 10''$; $\therefore$ sup. of comp. $= 180° - 46° 34' 10'' = 133° 25' 50''$.

**Ex. 13.** How many degrees are there in the complement of the supplement of an angle of 159°? Of 133° 15′ 25″?

**Solution.**

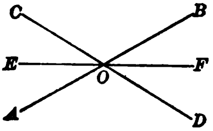
(1) Sup. $= 180^\circ - 159^\circ = 21^\circ$; $\therefore$ comp. of sup. $= 90^\circ - 21^\circ = 69^\circ$.

(2) Sup. $= 180^\circ - 133^\circ\,15'\,25'' = 46^\circ\,44'\,35''$; $\therefore$ comp. of sup. $= 90^\circ - 46^\circ\,44'\,35'' = 43^\circ\,15'\,25''$.

**Ex. 14.** The bisectors of two supplementary adjacent angles form an angle of 90°.

**Proof.** The required $\angle =$ one half the sum of the given $\angle$s $= \frac{1}{2}$ of $180^\circ = 90^\circ$.

**Ex. 15.** If a line drawn through the vertex of two vertical angles bisects one angle, it bisects the other.



**Data:** Vert. $\angle$s $AOC$ and $BOD$, and the bisector $EOF$ of $\angle AOC$.

**To prove** $EOF$ bisects $\angle BOD$.

**Proof.**

§ 59, $\angle COE = \angle DOF$, and $\angle AOE = \angle BOF$;

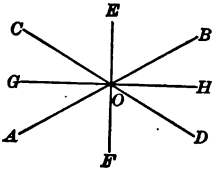
but $\angle COE = \angle AOE$; $\therefore$ Ax. 1, $\angle DOF = \angle BOF$;

that is, $EOF$ bisects $\angle BOD$.

**Ex. 16.** If one of the vertical angles formed by the intersection of two straight lines is 37°, what is the value of each of the other angles?

**Solution.** By § 59, the $\angle$ vertical to the given $\angle = 37^\circ$. Then, the sum of the other vert. $\angle$s $= 360^\circ - (37^\circ \times 2) = 286^\circ$, and each one $= \frac{1}{2}$ of $286^\circ$, or $143^\circ$.

**Ex. 17.** If lines are drawn to bisect the two pairs of vertical angles formed by two intersecting straight lines, the bisectors are perpendicular to each other.



**Data:** $EF$ and $GH$, the bisectors of the two pairs of vert. $\angle$s formed by $AB$ and $CD$ intersecting at $O$.

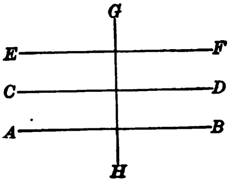
**To prove** $EF \perp GH$.

**Proof.** § 55, $\angle AOC + \angle COB = 2$ rt. $\angle$s;

but $\angle GOC = \frac{1}{2} \angle AOC$, and $\angle COE = \frac{1}{2} \angle COB$;

$\therefore$ $\angle GOC + \angle COE = 1$ rt. $\angle$; that is, $\angle GOE$ is a rt. $\angle$, and $EF \perp GH$.

**Ex. 18.** If two lines are drawn each parallel to $AB$, and another making an angle of 90° with $AB$, the third line is perpendicular to each of the other two lines.



**Data:** $EF$ and $CD$ each parallel to $AB$, and $GH$ making an angle of 90° with $AB$.

**To prove** $GH \perp EF$ and $CD$.

**Proof.** Data, $EF \parallel AB$; $\therefore$ §§ 26, 72, $GH \perp EF$. Similarly, $GH \perp CD$.

Ex. **19**. State and illustrate the differences between a plumb line, a perpendicular line, and a vertical line.

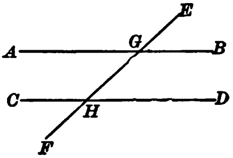
Consult any good dictionary.

Ex. **20**. Two parallel lines are cut by a third line, making one interior angle 35°. What is the value of the adjacent interior angle ?

**Solution.** The required $\angle$ is sup. of the given $\angle$; $\therefore$ the required $\angle = 180° - 35° = 145°$.

Ex. **21**. If two parallel lines are cut by a transversal, the sum of the two exterior angles on the same side of the transversal is equal to two right angles.

**Data**: Two parallel lines, $AB$ and $CD$, cut by the transversal $EF$ in $G$ and $H$ respectively.



**To prove** the ext. $\angle s$ on the same side of $EF$ equal to 2 rt. $\angle s$.

**Proof.** § 78, $\angle FGB + \angle EHD = 2$ rt. $\angle s$; but, § 76, $\angle FGB = \angle FHD$, and $\angle EHD = \angle EGB$; $\therefore$ $\angle FHD + \angle EGB = 2$ rt. $\angle s$.

Similarly, $\angle AGE + \angle CHF = 2$ rt. $\angle s$.

Ex. **22**. The straight lines $AB$ and $CD$ are cut by $EF$ in $G$ and $H$ respectively; angle $EHD = 38°$. What must be the value of the angle $EGB$ in order that $AB$ and $CD$ may be parallel ?

**Solution.** By § 77, $\angle EGB$ must be 38°.

Ex. **23**. A transversal cutting two parallel lines makes an interior angle of 50°. What is the value of the other interior angle on the same side of the transversal ?

**Solution.** The sum of the two int. $\angle s$ on the same side of the transversal $= 180°$; $\therefore$ the required $\angle = 180° - 50° = 130°$.

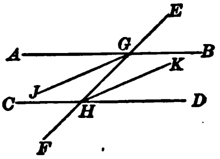
Ex. **24**. Two parallel lines are cut by a third line making one interior angle 35°. What is the value of each of the other interior angles ? How many degrees are there in the sum of the interior angles upon the same side of the transversal ?

**Solution.** By § 55, the given $\angle$ and the adj. int. $\angle = 180°$; $\therefore$ the adj. int. $\angle = 180° - 35° = 145°$; hence, § 73, the other int. $\angle s$ are 35° and 145° respectively.

By § 78, the sum of the int. $\angle s$ on the same side of the transversal $= 2$ rt. $\angle s = 180°$.

Ex. **25**. The bisectors of any two alternate interior angles, formed by two parallel lines cut by a transversal, are parallel.

**Data:** Two parallel lines, $AB$ and $CD$, cut by the transversal $EF$ in $G$ and $H$ respectively, and the lines $JG$ and $HK$, bisecting any two alt. int. $\angle$s, as $AGF$ and $EHD$, respectively.



**To prove** $JG \parallel HK$.

**Proof.** § 73, $\angle AGF = \angle EHD$; but $\angle JGF = \frac{1}{2}\angle AGF$, and $\angle EHK = \frac{1}{2}\angle EHD$; $\therefore$ Ax. 7, $\angle JGF = \angle EHK$; hence, § 75, $JG \parallel HK$.

Ex. **26.** The straight lines $AB$ and $CD$ are cut by $EF$ in $G$ and $H$ respectively; angle $EHD = 40°$. What must be the value of angle $AGF$, if $AB$ and $CD$ are parallel?

**Solution.** By § 75, $\angle AGF$ must be $40°$.

Ex. **27.** $AB$ and $CD$ are two lines cut in $G$ and $H$, respectively, by $EF$; $\angle BGF = 123°$, and $\angle GHD = 62°$. Are the lines $AB$ and $CD$ parallel?

**Solution.** If $AB \parallel CD$, § 78, $\angle BGF + \angle GHD = 180°$; but $\angle BGF + \angle GHD = 123° + 62° = 185°$; consequently, $AB$ is not parallel to $CD$.

Ex. **28.** If two lines are cut by a transversal and the sum of the two exterior angles on the same side of the transversal is equal to 180°, the lines are parallel.

**Proof.** § 76, the given $\angle$s are equal respectively to the int. $\angle$s which are not adjacent on the same side of the transversal; but the sum of the given $\angle$s $= 2$ rt. $\angle$s; $\therefore$ the sum of the int. $\angle$s on the same side of the transversal $= 2$ rt. $\angle$s.

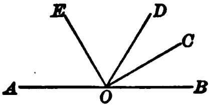
Hence, § 79, the lines are parallel.

Ex. **29.** Two parallel lines are cut by a transversal so that one exterior angle is 105°. How many degrees are there in the sum of each pair of alternate interior angles?

**Solution.** The int. $\angle$ adjacent to the given $\angle = 180° - 105° = 75°$, and its alt. int. $\angle = 75°$; $\therefore$ the sum of one pair of alt. int. $\angle$s $= 75° + 75° = 150°$. Each one of the other int. $\angle$s $= 180° - 75° = 105°$; $\therefore$ the sum of the second pair of alt. int. $\angle$s $= 105° + 105° = 210°$.

Ex. **30.** If the bisectors of two adjacent angles are perpendicular to each other, the angles are supplementary.

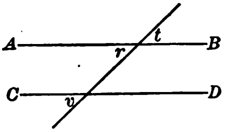
**Data:** Two adj. $\angle$s, $AOD$ and $BOD$, whose bisectors $EO$ and $CO$ are perpendicular to each other.



**To prove** $\angle$s $AOD$ and $BOD$ supplementary.

**Proof.** Data, § 26, $\angle EOD + \angle DOC = 1$ rt. $\angle$; then, Ax. 6, $\angle AOD + \angle BOD = 2$ rt. $\angle$s; that is, § 32, $\angle$s $AOD$ and $BOD$ are supplementary.

**Ex. 31.** Two lines are cut by a transversal. If the alternate exterior angles are equal, the lines are parallel.



**Data:** Two lines, $AB$ and $CD$, such that when cut by a transversal the alt. ext. $\angle$s, as $t$ and $v$, are equal.

**To prove** $AB \parallel CD$.

**Proof.** Data, $\angle v = \angle t$; but, § 59, $\angle r = \angle t$; $\therefore \angle v = \angle r$, and § 77, $AB \parallel CD$.

Ex. **32.** The straight lines $AB$, $CD$, and $EF$ are cut in $G$, $H$, and $J$, respectively, by $KL$; angle $KGB = 37°$; angle $KHC = 149°$; angle $FJL = 143°$. (1) Are the lines $AB$ and $CD$ parallel? (2) $AB$ and $EF$? (3) $CD$ and $EF$?

**Proof.** (1) If $AB \parallel CD$, § 76, $\angle KGB = \angle KHD$; but these angles are not equal, for $\angle KGB = 37°$ and $\angle KHD = 180° - \angle KHC = 180° - 149° = 31°$; hence, $AB$ and $CD$ are not parallel.

(2) $\angle KJF = 180° - \angle FJL = 180° - 143° = 37°$; but $\angle KGB = 37°$;

$\therefore \angle KGB = \angle KJF$; hence, § 77, $AB \parallel EF$.

(3) If $CD \parallel EF$, § 76, $\angle KHC = \angle KJE$; but these angles are not equal, for $\angle KHC = 149°$, and, § 59, $\angle KJE = \angle FJL = 143°$; hence, $CD$ and $EF$ are not parallel.

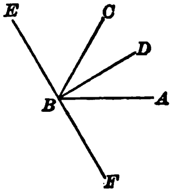
**Ex. 33.** Two intersecting straight lines cannot both be parallel to the same straight line.

**Proof.** If the lines could be parallel to the same straight line, § 80, they would be parallel to each other, but this is impossible, for, § 63, parallel lines cannot intersect; consequently, two intersecting straight lines cannot both be parallel to the same straight line.

Ex. **34.** The angle formed by the bisectors of two complementary adjacent angles is an angle of 45°.

**Proof.** The required $\angle$ = one half the sum of the given $\angle$s = $\frac{1}{2}$ of $90° = 45°$.

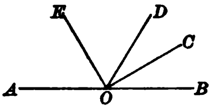
Ex. **35.** If the line $BD$ bisects the angle $ABC$, and $EF$ is drawn through $B$ perpendicular to $BD$, the angles $CBE$ and $ABF$ are equal.



**Proof.** $\angle ABF = 1$ rt. $\angle - \angle ABD$, and $\angle CBE = 1$ rt. $\angle - \angle CBD$; but $\angle ABD = \angle CBD$; $\therefore \angle ABF = \angle CBE$.

Ex. **36.** If a straight line is perpendicular to the bisector of an angle at the vertex, it bisects the supplementary adjacent angle formed by producing one side of the given angle through the vertex.

**Data**: $OC$ the bisector of $\angle BOD$; $EO \perp OC$ at $O$; and the $\angle DOA$ formed by producing $BO$.

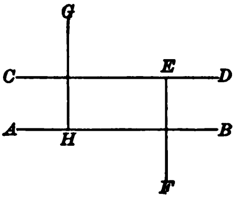


**To prove** $EO$ bisects $\angle DOA$.

**Proof.** § 56, $\angle BOC + \angle COD + \angle DOE + \angle EOA = 2$ rt. $\angle$s; data, Ax. 9, $\angle COD + \angle DOE = 1$ rt. $\angle$; $\therefore$ Ax. 3, $\angle BOC + \angle EOA = 1$ rt. $\angle$.

Then, $\angle DOE = 1$ rt. $\angle - \angle COD$, and $\angle EOA = 1$ rt. $\angle - \angle BOC$; but, data, $\angle COD = \angle BOC$; $\therefore \angle DOE = \angle EOA$; that is, $EO$ bisects $\angle DOA$.

Ex. **37**. If two straight lines are perpendicular each to one of two parallel lines, they are parallel.



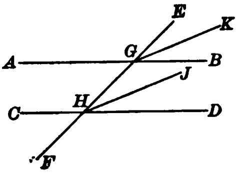
**Data**: Parallel lines, $AB$ and $CD$; $GH \perp AB$; and $EF \perp CD$.

**To prove** $EF \parallel GH$.

**Proof.** Data, $EF \perp CD$; $\therefore$ § 72, $EF \perp AB$; but, data, $GH \perp AB$; hence, § 71, $EF \parallel GH$.

Ex. **38**. The bisectors of any two corresponding angles formed by parallel lines cut by a transversal, are parallel.

**Data**: Two parallel lines, $AB$ and $CD$, cut by the transversal $EF$ in $G$ and $H$, respectively, and the lines $GK$ and $HJ$, bisecting any two corresponding $\angle$s, as $EGB$ and $EHD$, respectively.

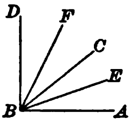


**To prove** $GK \parallel HJ$.

**Proof.** § 76, $\angle EGB = \angle EHD$; $\therefore \frac{1}{2} \angle EGB = \frac{1}{2} \angle EHD$; that is, $\angle EGK = \angle EHJ$; hence, § 77, $GK \parallel HJ$.

Ex. **39**. If the bisectors of two adjacent angles form an angle of 45°, the angles are complementary.

**Data**: The bisectors, $EB$ and $FB$, of the adj. $\angle$s $ABC$ and $CBD$, forming an angle of 45°.



**To prove** $\angle$s $ABC$ and $CBD$ complementary.

**Proof.** Data, $\angle EBC + \angle FBC = 45°$; then, Ax. 6, $\angle ABC + \angle CBD = 90°$; that is, § 31, $\angle$s $ABC$ and $CBD$ are complementary.

Ex. **40**. Two angles are supplementary, and the greater is five times the less. How many degrees are there in each angle?

**Solution.** The sum of the $\angle$s $= 6$ times the less $= 180°$; then, the less $\angle = \frac{1}{6}$ of $180° = 30°$, and the greater $\angle = 180° - 30° = 150°$.

Ex. **41**. Two angles are complementary, and the greater is five times the less. How many degrees are there in each angle?

**Solution.** The sum of the $\angle$s $= 6$ times the less $= 90°$; then, the less $\angle = \frac{1}{6}$ of $90° = 15°$, and the greater $\angle = 90° - 15° = 75°$.

Ex. **42.** Two parallel straight lines are cut by a transversal so that one of the two interior angles on one side of the transversal is eleven times the other. How many degrees are there in each of the exterior angles?

**Solution.** § 78, the sum of the two int. $\angle$s on the same side of the transversal $= 12$ times the less $= 180°$; then, the less $\angle = \frac{1}{12}$ of $180° = 15°$, and the greater $\angle = 180° - 15° = 165°$; consequently, §§ 59, 76, there are two ext. $\angle$s of 15° each and two of 165° each.

PROP. XVIII, page 40. (1) When $AB = DE$, $BC = EF$, and angle $B =$ angle $E$.

**Proof.** Place $\triangle ABC$ upon $\triangle DEF$, $AB$ coinciding with $DE$.

Data, $\angle B = \angle E$, hence $BC$ will take the direction of $EF$; and since $BC = EF$, the point $C$ will fall upon the point $F$.

Since the point $A$ falls upon $D$ and the point $C$ upon $F$,

$AC$ will coincide with $DF$.

Then, $\triangle$s $ABC$ and $DEF$ coincide in all their parts.

Hence, § 36, $\triangle ABC = \triangle DEF$.

(2) When $AC = DF$, $BC = EF$, and angle $C =$ angle $F$.

**Proof.** Place $\triangle ABC$ upon $\triangle DEF$, $AC$ coinciding with $DF$.

Data, $\angle C = \angle F$; hence, $CB$ will take the direction of $FE$, and since $BC = EF$, the point $B$ will fall upon the point $E$.

Since the point $A$ falls upon the point $D$, and the point $B$ upon $E$,

$AB$ will coincide with $DE$.

Then, $\triangle$s $ABC$ and $DEF$ coincide in all their parts.

Hence, § 36, $\triangle ABC = \triangle DEF$.

PROP. XIX, page 41. (1) When angle $C =$ angle $F$, angle $B =$ angle $E$, and $BC = EF$.

**Proof.** Place $\triangle ABC$ upon $\triangle DEF$, $BC$ coinciding with $EF$.

Data, $\angle C = \angle F$; hence, $CA$ will take the direction of $FD$, and the point $A$ will fall upon $FD$, or upon $FD$ produced.

Also, data, $\angle B = \angle E$; hence, $BA$ will take the direction of $ED$, and the point $A$ will fall upon $ED$, or upon $ED$ produced.

Since the point $A$ falls upon each of the lines $FD$ and $ED$, it must fall upon their point of intersection $D$.

Then, $\triangle$s $ABC$ and $DEF$ coincide in all their parts.

Hence, § 36, $\triangle ABC = \triangle DEF$.

(2) When angle $A =$ angle $D$, angle $C =$ angle $F$, and $AC = DF$.

**Proof.** Place $\triangle ABC$ upon $\triangle DEF$, $AC$ coinciding with $DF$.

Data, $\angle A = \angle D$; hence, $AB$ will take the direction of $DE$, and the point $B$ will fall upon $DE$, or upon $DE$ produced.

Also, data, $\angle C = \angle F$; hence, $CB$ will take the direction of $FE$, and the point $B$ will fall upon $FE$, or upon $FE$ produced.

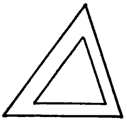
Since the point $B$ falls upon each of the lines $DE$ and $FE$, it must fall upon their point of intersection $E$.

Then, $\triangle$s $ABC$ and $DEF$ coincide in all their parts.

Hence, § 36, $\triangle ABC = \triangle DEF$.

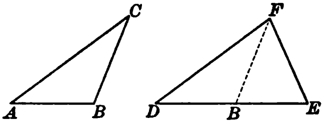
**Ex. 43.** Are two triangles equal, if the three angles of one are equal to the three angles of the other, each to each?

No, not necessarily. The accompanying figure represents two triangles whose angles are equal, each to each, but the triangles are evidently unequal, for their parts will not coincide if one triangle is applied to the other.



**Ex. 44.** Can two triangles, having two sides and an angle of one respectively equal to two sides and an angle of the other, be unequal?

Yes, for example, if two triangles have two sides of one equal to two sides of the other, each to each, and an angle opposite one of these sides in one triangle equal to the corresponding angle in the other, and if the angles opposite the other equal sides are one obtuse and the other acute, then the triangles are unequal.



**Data:** $\triangle$s $ABC$ and $DEF$ in which $AC = DF$, $BC = EF$, $\angle A$ (opp. $BC$) $= \angle D$ (opp. $EF$), $\angle B$ obtuse and $\angle E$ acute.

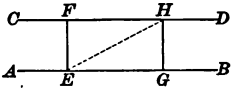
**To prove** $\triangle$s $ABC$ and $DEF$ unequal.

**Proof.** Apply $\triangle ABC$ to $\triangle DEF$ so that the equal sides $AC$ and $DF$ shall coincide.

Data, $\angle A = \angle D$; $\therefore$ $AB$ falls upon $DE$, but $\angle B$ cannot coincide with $\angle E$, for, data, they are unequal, one being obtuse and the other acute; consequently, the $\triangle$s cannot be made to coincide in all their parts and are, therefore, unequal, for, § 36, equal $\triangle$s can be made to coincide in all their parts.

**Ex. 45.** Parallel lines are everywhere equally distant.

**Data:** Two parallel lines, $AB$ and $CD$, and the ⊥s $EF$ and $GH$ representing the distance between them at any two points.



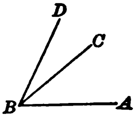
**To prove** $EF = GH$.

**Proof.** Draw $EH$. § 71, $EF \parallel GH$.

Then, in $\triangle$s $EFH$ and $EGH$, $EH$ is common, § 73, $\angle EHF = \angle HEG$, and $\angle FEH = \angle EHG$;

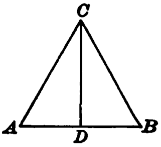
$\therefore$ § 102, $\triangle EFH = \triangle EGH$, and $EF = GH$.

**Ex. 46.** Can two angles which are not adjacent have a common vertex and a common side?



Yes. In the accompanying figure $\angle$s *ABC* and *ABD* have a common vertex *B*, and a common side *AB*; but, § 25, they are not adjacent angles.

Ex. **47.** If in an equilateral triangle a line is drawn from the vertex to the middle point of the base, the triangles thus formed are equal.



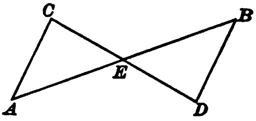
**Data:** Equilateral $\triangle ABC$ and the line *CD* from the vertex *C* to the middle point of *AB*.

**To prove** $\triangle ADC = \triangle BDC$.

**Proof.** Data, *C* and *D* are each equidistant from *A* and *B*; $\therefore$ § 106, $CD \perp AB$, and, § 26, $\angle$s *ADC* and *BDC* are rt. $\angle$s.

Then, in $\triangle$s *ADC* and *BDC*, data, $AD = BD$, *CD* is common, and, § 52, $\angle ADC = \angle BDC$; $\therefore$ § 100, $\triangle ADC = \triangle BDC$.

Ex. **48.** If two lines bisect each other, the lines joining their opposite extremities are parallel.

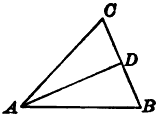


**Data:** Two lines, *AB* and *CD* bisecting each other at *E*.

**To prove** $AC \parallel DB$.

**Proof.** Data, $AE = BE$, $CE = DE$, and, § 59, $\angle AEC = \angle DEB$; $\therefore$ § 100, $\triangle AEC = \triangle DEB$, and $\angle A = \angle B$; hence, § 75, $AC \parallel DB$.

Ex. **49.** If two sides of a triangle are equal, and a line is drawn bisecting their included angle and intersecting the third side, the segments of the third side are equal.

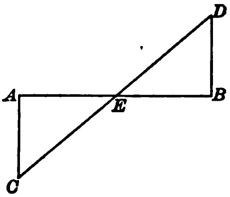


**Data:** $\triangle ABC$ in which $AB = AC$, and *AD* bisects $\angle A$ and meets *BC* in *D*.

**To prove** $BD = CD$.

**Proof.** In $\triangle$s *ABD* and *ACD*, data, $AB = AC$, *AD* is common, and $\angle BAD = CAD$; $\therefore$ § 100, $\triangle ABD = \triangle ACD$, and $BD = CD$.

Ex. **50.** If perpendiculars are erected at the extremities of a line and terminate in any bisector of the line that is not perpendicular to the line, the perpendiculars are equal.

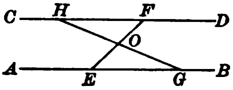


**Data:** *AB* bisected at *E* by *CD*; also *AC* and $BD \perp AB$ and meeting *CD* in *C* and *D* respectively.

**To prove** $AC = BD$.

**Proof.** Data, $AE = BE$, § 59, $\angle AEC = \angle BED$, and, § 52, $\angle CAE = \angle EBD$; $\therefore$ § 102, $\triangle AEC = \triangle BED$, and $AC = BD$.

Ex. **51.** If through the middle point of a straight line terminating in two parallel lines, a second straight line is drawn also terminating in the parallels, the parts of the second line are equal.



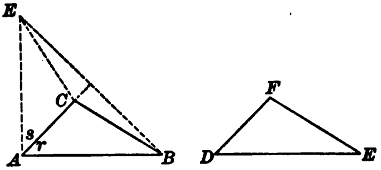
**Data:** $AB \parallel CD$, and $EF$ terminating in $AB$ and $CD$; also any other line as $HG$ through $O$, the middle point of $EF$, terminating in $AB$ and $CD$.

**To prove** $OG = OH$.

**Proof.** Data, $EO = FO$, § 59, $\angle EOG = \angle HOF$, and, § 73, $\angle OEG = \angle OFH$; $\therefore$ § 102, $\triangle EOG = \triangle FOH$, and $OG = OH$.

Prop. XXI, page 44. (1) Prove by placing the triangle so that $DF$ will coincide with $AC$.

**Proof.** Place $\triangle DEF$ in the position $AEC$, so that the equal sides, $DF$ and $AC$, coincide, and the vertex $E$ falls opposite $B$. Draw $BE$ and produce $AC$ to meet $BE$.

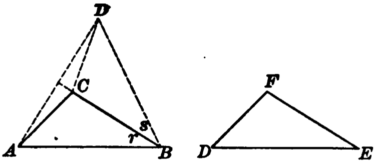


Data, $AB = AE$, and $CB = CE$; $\therefore$ $A$ and $C$ are each equidistant from $B$ and $E$; hence, § 106, $AC \perp BE$ at its middle point, and, § 105, $\angle r = \angle s$.

In $\triangle$s $ABC$ and $AEC$, $AB = AE$, $AC$ is common, and $\angle r = \angle s$; $\therefore$ § 100, $\triangle ABC = \triangle AEC$; that is, $\triangle ABC = \triangle DEF$.

(2) Prove by placing the triangle so that $EF$ will coincide with $BC$.

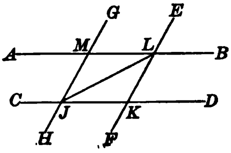
**Proof.** Place $\triangle DEF$ in the position $BDC$, so that the equal sides, $EF$ and $BC$, coincide, and the vertex $D$ falls opposite $A$. Draw $AD$ and produce $BC$ to meet $AD$.



Data, $AB = BD$, and $AC = CD$; $\therefore$ $B$ and $C$ are each equidistant from $A$ and $D$; hence, § 106, $BC \perp AD$ at its middle point, and, § 105, $\angle r = \angle s$.

In $\triangle$s $ABC$ and $BDC$, $AB = BD$, $BC$ is common, and $\angle r = \angle s$; $\therefore$ § 100, $\triangle ABC = \triangle BDC$; that is, $\triangle ABC = \triangle DEF$.

Ex. **52.** If two parallel lines intersect two parallel lines, and a line is drawn joining two opposite points of intersection, the triangles thus formed are equal.

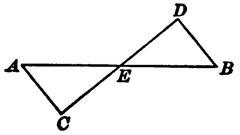


**Data:** Parallels $AB$ and $CD$ intersecting parallels $EF$ and $GH$ in $L$, $M$, $K$, and $J$; also line $JL$.

**To prove** $\triangle JKL = \triangle JML$.

**Proof.** § 73, $\angle LJK = \angle JLM$, $\angle JLK = \angle LJM$, and $JL$ is common; $\therefore$ $\triangle JKL = \triangle JML$.

Ex. **53**. If perpendiculars are drawn from the extremities of a line to any line that bisects it and is not perpendicular to it, the perpendiculars are equal.



**Data**: $AB$ bisected at $E$ by $CD$; also $AC$ and $BD \perp CD$ meeting $CD$ in $C$ and $D$ respectively.

**To prove** $AC = BD$.

**Proof.** Data, $AC \perp CD$, and $BD \perp CD$; $\therefore$ § 71, $AC \parallel BD$, and, § 73, $\angle A = \angle B$, also, § 59, $\angle AEC = \angle BED$, and, data, $AE = BE$; $\therefore$ § 102, $\triangle AEC = \triangle BED$, and $AC = BD$.

Ex. **54**. If the line $BD$ is the bisector of the angle $ABC$ whose sides are equal, and lines are drawn from any point of $BD$, as $E$, to $A$ and $C$, $AE$ and $CE$ are equal.

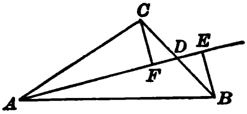
**Proof.** In $\triangle$s $ABE$ and $CBE$, $BE$ is common, data, $AB = BC$, and $\angle ABE = \angle CBE$; $\therefore$ § 100, $\triangle ABE = \triangle CBE$, and $AE = CE$.

Ex. **55**. If in a triangle $ABC$, angle $A$ equals angle $B$, and a line parallel to $AB$ intersects $AC$ in $D$ and $BC$ in $E$, the angles $ADE$ and $BED$ are equal.

**Proof.** § 76, $\angle CDE = \angle A$, and $\angle CED = \angle B$; but, data, $\angle A = \angle B$; $\therefore$ Ax. 1, $\angle CDE = \angle CED$.

Now, § 55, $\angle ADE$ is sup. of $\angle CDE$, and $\angle BED$ is sup. of $\angle CED$; $\therefore$ § 54, $\angle ADE = \angle BED$.

Ex. **56**. If $D$ is the middle point of the side $BC$ of the triangle $ABC$, and $BE$ and $CF$ are perpendiculars from $B$ and $C$ to $AD$, or $AD$ produced, $BE$ and $CF$ are equal.



**Proof.** Data, $BD = CD$, § 59, $\angle BDE = \angle CDF$, and since, § 71, $BE$ and $CF$ are parallel, § 73, $\angle DBE = \angle DCF$; $\therefore$ § 102, $\triangle BDE = \triangle CDF$, and $BE = CF$.

Ex. **57**. May a triangle be formed whose angles are (1) 93°, 40°, and 61° respectively? (2) 98°, 24°, and 58°? (3) 57°, 49°, and 74°?

(1) No, for, § 110, the sum of the $\angle$s of a $\triangle = 180°$; but $93° + 40° + 61° = 194°$.

(2) Yes, for $98° + 24° + 58° = 180°$.

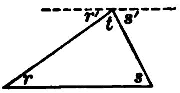
(3) Yes, for $57° + 49° + 74° = 180°$.

Ex. **58**. Two angles of a triangle are together equal to 76°. What is the value of the third angle?

**Solution.** § 110, the sum of the $\angle$s of a $\triangle = 180°$; $\therefore$ the required $\angle = 180° - 76° = 104°$.

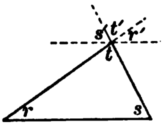
Ex. **59**. Show by each of the following figures that the sum of the three angles of a triangle is equal to two right angles, assuming that the construction lines are drawn as they appear to be drawn.

**Proof.** (1) § 73, $\angle r = \angle r'$, and $\angle s = \angle s'$;
$\therefore$ $\angle r + \angle t + \angle s = \angle r' + \angle t + \angle s'$;
but, § 56, $\angle r' + \angle t + \angle s' = 2$ rt. $\angle$s;
$\therefore$ $\angle r + \angle t + \angle s = 2$ rt. $\angle$s.



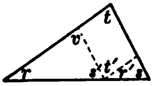
(1)

(2) § 76, $\angle r = \angle r'$, and $\angle s = \angle s'$,
and, § 59, $\angle t = \angle t'$;
$\therefore$ $\angle r + \angle t + \angle s = \angle r' + \angle t' + \angle s'$;
but, § 56, $\angle r' + \angle t' + \angle s' = 2$ rt. $\angle$s;
$\therefore$ $\angle r + \angle t + \angle s = 2$ rt. $\angle$s.



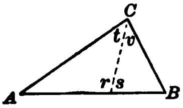
(2)

(3) § 76, $\angle r = \angle r'$, $\angle s = \angle s'$,
and, §§ 76, 73, $\angle t = (\angle v) = \angle t'$;
$\therefore$ $\angle r + \angle t + \angle s = \angle r' + \angle t' + \angle s'$;
but, § 56, $\angle r' + \angle t' + \angle s' = 2$ rt. $\angle$s;
$\therefore$ $\angle r + \angle t + \angle s = 2$ rt. $\angle$s.



(3)

(4) § 115, $\angle s = \angle A + \angle t$,
and $\angle r = \angle B + \angle v$;
$\therefore$ $\angle s + \angle r = \angle A + \angle B + \angle t + \angle v$
$= \angle A + \angle B + \angle C$;
but, § 56, $\angle s + \angle r = 2$ rt. $\angle$s;
$\therefore$ $\angle A + \angle B + \angle C = 2$ rt. $\angle$s.



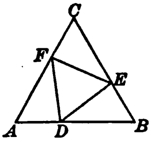
(4)

Ex. **60.** If equal distances from the vertices of an equilateral triangle are laid off on its sides in the same order, the lines joining these points form an equilateral triangle.

**Data:** Equilateral $\triangle ABC$; the equal distances $AD$, $BE$, and $CF$ laid off on its sides from the vertices; and the lines joining the points $D$, $E$, and $F$.

**To prove** $\triangle DEF$ equilateral.

**Proof.** Data, $AB = BC = AC$, and $AD = BE = CF$; $\therefore$ Ax. 3, $DB = CE = AF$. Also, § 117, $\angle A = \angle B = \angle C$; $\therefore$ § 100, $\triangle ADF = \triangle DBE = \triangle ECF$, and, § 108, $DF = DE = EF$; that is, $\triangle DEF$ is equilateral.

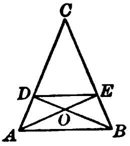


Ex. **61.** If from the extremities of the base of an isosceles triangle perpendiculars are drawn to the opposite sides, and the points where the perpendiculars meet the opposite sides are joined by a straight line, this line is parallel to the base.

**Data:** Isos. $\triangle ABC$; $AE \perp BC$, and $BD \perp AC$, at $E$ and $D$ respectively; also the line $DE$.

**To prove** $DE \parallel AB$.

**Proof.** In rt. $\triangle$s $ABE$ and $ABD$, $AB$ is common, and, § 116, $\angle ABE = \angle DAB$; $\therefore$ § 114, $\triangle ABE = \triangle ABD$, and $\angle EAB = \angle ABD$; hence, Ax. 3, $\angle DAE = \angle DBE$, and, § 108, $AD = BE$; $\therefore$ § 114, $\triangle AOD = \triangle BOE$, and $DO = OE$; $\therefore$ § 116, $\angle OED = \angle ODE$. Similarly, $\angle OAB = \angle OBA$.



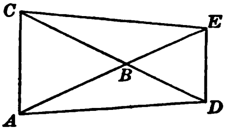
Then, § 110, $2\angle OED + \angle DOE = 2$ rt. $\angle$s, and $2\angle OAB + \angle AOB = 2$ rt. $\angle$s; hence, $2\angle OED + \angle DOE = 2\angle OAB + \angle AOB$; but, § 59, $\angle DOE = \angle AOB$; $\therefore$ Ax. 3, $2\angle OED = 2\angle OAB$, or $\angle OED = \angle OAB$; hence, § 75, $DE \parallel AB$.

**Ex. 62.** The base of an isosceles triangle is 6 inches and the angle opposite is 60°. How many degrees are there in each of the base angles? What is the length of each of the other two sides?

**Solution.** The sum of the base $\angle$s $= 180° - 60° = 120°$; $\therefore$ each base $\angle = \frac{1}{2}$ of $120° = 60°$.

Since the $\triangle$ is equiangular, § 119, it is also equilateral and each side is 6 in. long.

**Ex. 63.** The lines joining the extremities of the bases of two opposite, or vertical, isosceles triangles are equal.



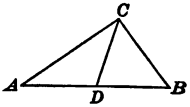
**Data:** Vert. isos. $\triangle$s $ABC$ and $DBE$, and the lines $AD$ and $CE$ joining the extremities of their bases.

**To prove** $AD = CE$.

**Proof.** $AB = BC$, $BD = BE$, and, § 59, $\angle ABD = \angle CBE$; $\therefore$ § 100, $\triangle ABD = \angle CBE$, and $AD = CE$.

**Ex. 64.** If the median line from the vertex to the base of a certain triangle is equal to one half the base, the vertical angle is a right angle.

**Data:** $\triangle ABC$ such that the median $CD = \frac{1}{2} AB$.



**To prove** $\angle ACB$ a rt. $\angle$.

**Proof.** Data, $CD = AD = DB$;

$\therefore$ § 116, $\angle A = \angle ACD$, and $\angle B = \angle BCD$;

hence, $\angle A + \angle B = (\angle ACD + \angle BCD) = \angle ACB$;

but $\angle A + \angle B + \angle ACB = 180°$;

$\therefore$ $\angle ACB = \frac{1}{2}$ of 180°; that is, $\angle ACB$ is a rt. $\angle$.

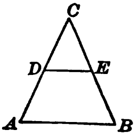
**Ex. 65.** (1) May a triangle be formed with lines 4, 2, and 3 inches long? (2) With lines 6, 1, and 2 inches long? (3) 5, 2, and 3 inches long?

(1) Yes, for the sum of any two sides would be greater than the third, which, by § 125, must be true for the sides of a $\triangle$.

(2) No, for 1 in. + 2 in. is not greater than 6 in.

(3) No, for 2 in. + 3 in. is not greater than 5 in.

**Ex. 66.** If a line is drawn joining the middle points of the equal sides of an isosceles triangle, an isosceles triangle is formed.

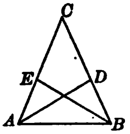


**Data:** An isos. $\triangle ABC$ and the line $DE$ joining the middle points of the equal sides $AC$ and $BC$.

**To prove** $\triangle DEC$ isosceles.

**Proof.** Data, $DC = \frac{1}{2} AC$, $EC = \frac{1}{2} BC$, and $AC = BC$; $\therefore$ Ax. 7, $DC = EC$, and, § 90, $\triangle DEC$ is isosceles.

**Ex. 67.** If the bisectors of the base angles of an isosceles triangle are produced to the opposite sides, they are equal.



**Data:** The bisectors, $AD$ and $BE$, of the base angles $A$ and $B$, of the isos. $\triangle ABC$ produced to meet $BC$ and $AC$ in $D$ and $E$ respectively.

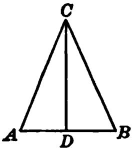
**To prove** $AD = BE$.

**Proof.** In $\triangle$s $ABD$ and $ABE$, § 116, $\angle ABD = \angle BAE$, Ax. 7, $\angle BAD = \angle ABE$, and $AB$ is common; $\therefore$ § 102, $\triangle ABD = \triangle ABE$, and $AD = BE$.

**Ex. 68.** The sum of the two angles at the base of an isosceles triangle is 64°. What is the value of each angle of the triangle?

**Solution.** Each base $\angle = \frac{1}{2}$ of $64° = 32°$. The vert. $\angle = 180° - 64° = 116°$.

**Ex. 69.** If the straight line which joins the vertex of a triangle with the middle point of the base is perpendicular to the base, the triangle is isosceles.



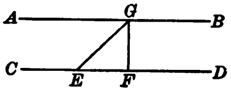
**Data:** $\triangle ABC$, and the line $CD$ from the vertex $C$ perpendicular to $AB$ at its middle pt.

**To prove** $\triangle ABC$ isosceles.

**Proof.** Data, $CD \perp AB$ at its middle pt.; then, § 103, $C$ is equidistant from $A$ and $B$; that is, $AC = BC$.

Hence, § 90, $\triangle ABC$ is isosceles.

**Ex. 70.** The perpendicular distance between two parallel lines is 20 inches, and a line is drawn across the parallels making an angle of 45° with the perpendicular at its upper extremity. What distance does this line cut off from the foot of the perpendicular?

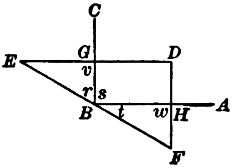


**Solution.** Let $FG$, the perpendicular between the parallels $AB$ and $CD$, be 20 in. long.

Let $EG$ make an $\angle$ of 45° with $FG$ at $G$, and meet $CD$ in $E$.

Then, $\triangle EFG$ is a rt. $\triangle$, and, § 111, $\angle GEF = 90° - 45° = 45° = \angle EGF$, and, § 118, $EF = FG = 20$ in.

**Ex. 71.** If from a point within a right angle perpendiculars are drawn to the sides containing the right angle, and each perpendicular is produced its own length, the line joining the extremities of the produced lines and the vertex of the right angle is a straight line.



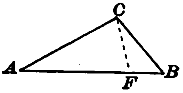
**Data:** Rt. $\angle ABC$; a pt. within, as $D$; $DG$ and $DH$ $\perp$s to $BC$ and $AB$, respectively, each produced its own length to $E$ and $F$, respectively; and the line $EBF$.

**To prove** $EBF$ a straight line.

**Proof.** § 71, $DE \parallel AB$, and $DF \parallel BC$; $\therefore$ data, and Ex. 45, $EG = (GD) = BH$, $BG = (DH) = FH$, and, § 52, $\angle v = \angle w$; $\therefore$ § 100, $\triangle EBG = \triangle BFH$, and $\angle r = \angle F$.

Now, § 111, $\angle t + \angle F = 1$ rt. $\angle$; $\therefore$ $\angle t + \angle r = 1$ rt. $\angle$;
but, data, $\angle s = 1$ rt. $\angle$; $\therefore$ Ax. 2, $\angle t + \angle s + \angle r = 2$ rt. $\angle$s,
or, Ax. 9, $\angle t + \angle ABE = 2$ rt. $\angle$s; hence, § 58, $EBF$ is a straight line.

**Prop. XXVIII**, page 52. Prove $\angle ACB$ greater than $\angle B$ when $AB$ is greater than $AC$.



**Proof.** On $AB$ take $AF = AC$, and draw $CF$.

Ax. 8, $\angle ACB$ is greater than $\angle ACF$;
but, § 116, $\angle ACF = \angle AFC$;
$\therefore$ $\angle ACB$ is greater than $\angle AFC$;
but since, § 115, $\angle AFC = \angle B + \angle BCF$, $\angle AFC$ is greater than $\angle B$.

Then, $\angle ACB$, the $\angle$ opposite $AB$, is greater than $\angle B$, the angle opposite $AC$.

**Ex. 72.** The angles of a scalene triangle are unequal.

**Proof.** § 89, the sides of a scalene $\triangle$ are unequal; $\therefore$ § 126, the angles opposite any two are unequal; hence, the three $\angle$s are unequal.

**Ex. 73.** What is the value of the angle formed by the bisectors of the acute angles of a right triangle?

**Solution.** The bisectors and the hypotenuse form a $\triangle$, the sum of two of whose $\angle$s $= \frac{1}{2}$ the sum of the acute $\angle$s of the rt. $\triangle = \frac{1}{2}$ of $90° = 45°$.

Then, the third $\angle$, which is the required $\angle$, $= 180° - 45° = 135°$.

**Ex. 74.** How many degrees are there in an angle of an equilateral triangle?

**Solution.** § 117, the equilateral $\triangle$ is equiangular;

$\therefore$ each $\angle = \frac{1}{3}$ of $180° = 60°$.

**Ex. 75.** How many degrees are there in each of the equal angles of an isosceles triangle, the angle at the vertex being 35° 50′?

**Solution.** The sum of the equal $\angle$s $= 180° - 35°\,50' = 144°\,10'$;

$\therefore$ each of the equal $\angle$s $= \frac{1}{2}$ of $144°\,10' = 72°\,5'$.

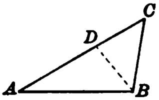
**Ex. 76.** In an isosceles triangle one base angle is 35°. What is the value of the vertical angle?

**Solution.** Since the base $\angle$s are equal, their sum $= 35° \times 2 = 70°$, and the vert. $\angle = 180° - 70° = 110°$.

**Ex. 77.** $AD$ is the bisector of a base angle of the isosceles triangle $ABC$, the bisector meeting the side $BC$ in $D$; the vertical angle $C$ is 28°. How many degrees are there in angle $ADC$?

**Solution.** $\angle B + \angle CAB = 180° - 28° = 152°$; $\therefore\ \angle CAB = \frac{1}{2}$ of $152° = 76°$, and, data, $\angle CAD = \frac{1}{2}\angle CAB = \frac{1}{2}$ of $76° = 38°$.
Then, $\angle ADC = 180° - (\angle C + \angle CAD) = 180° - (28° + 38°) = 114°$.

Prop. XXIX, page 53. Prove that $AC$ is greater than $AB$ when $\angle ABC$ is greater than $\angle C$.



**Proof.** Draw $BD$ so that $\angle CBD = \angle C$.
Then, § 118, $DC = BD$.
In $\triangle ABD$, § 125, $AD + BD > AB$;
$\therefore$ substituting $DC$ for its equal $BD$,

$$AD + DC > AB, \text{ or } AC > AB.$$

Ex. **78**. The angles $A$, $B$, and $C$ of the triangle $ABC$ are 40°, 60°, and 80°, respectively. (1) How do $AC$ and $AB$ compare in length? (2) $AB$ and $BC$? (3) $AC$ and $BC$?

(1) $AB$ is greater than $AC$, because, § 127, it is opposite a greater angle.

(2) $AB$ is greater than $BC$, because, § 127, it is opposite a greater angle.

(3) $AC$ is greater than $BC$, because, § 127, it is opposite a greater angle.

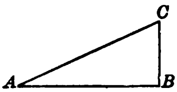
Ex. **79**. $CD$ bisects the base of an isosceles triangle $ABC$, a base angle of which is 55°. How many degrees are there in angle $ADC$? In angle $CDB$? In angle $ACD$? In angle $DCB$?

**Solution.** § 107, $\triangle ADC = \triangle BDC$; $\therefore\ \angle ADC = \angle CDB$, and, § 26, $CD \perp AB$; consequently, $\angle ADC = 90°$, and $\angle CDB = 90°$.

$$\angle ACD = 180° - (\angle A + \angle ADC) = 180° - (55° + 90°) = 35°.$$

Similarly, $\angle DCB = 35°$.

Ex. **80**. If one angle of a triangle is equal to the sum of the other two, it is a right angle, and the triangle is a right triangle.



**Data**: $\triangle ABC$ in which $\angle B = \angle A + \angle C$.

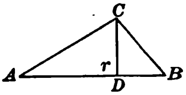
**To prove** $\triangle ABC$ a rt. $\triangle$.

**Proof.** § 110, $\angle A + \angle C + \angle B = 2$ rt. $\angle$s; but, data, $\angle A + \angle C = \angle B$; $\therefore$ substituting, $\angle B + \angle B = 2$ rt. $\angle$s, or $2\angle B = 2$ rt. $\angle$s, and $\angle B =$ a rt. $\angle$; hence, $\triangle ABC$ is a rt. $\triangle$.

Ex. **81**. $AD$ is perpendicular to $BC$, one of the equal sides of the isosceles triangle $ABC$ whose vertical angle is 30°. How many degrees are there in each of the angles $CAD$, $DAB$, and $ABC$?

**Solution.** $\angle CAD = 180° - (\angle ADC + \angle ACD) = 180° - (90° + 30°) = 60°$.
$\angle CAB + \angle B = 180° - 30° = 150°$; then, $\angle CAB = \frac{1}{2}$ of $150° = 75°$;
$\therefore\ \angle DAB = \angle CAB - \angle CAD = 75° - 60° = 15°$. $\angle ABC = \angle CAB = 75°$.

**Ex. 82.** The perpendicular let fall from the vertex to the base of a triangle divides the vertical angle into two angles whose difference is equal to the difference of the base angles of the triangle.



**Data:** $\triangle ABC$ and the perpendicular $CD$ from the vertex $C$ to the base $AB$, $\angle ACD$ being greater than $\angle BCD$.

**To prove** $\angle ACD - \angle BCD = \angle B - \angle A$.

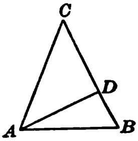
**Proof.** § 111, $\angle ACD = 90° - \angle A$, and $\angle BCD = 90° - \angle B$; $\therefore$ subtracting, $\angle ACD - \angle BCD = \angle B - \angle A$.

**Ex. 83.** $ABC$ is a triangle. Angle $A = 60°$, angle $B = 40°$. The bisector of angle $A$ is produced until it cuts the side $BC$. How many degrees are there in each angle thus formed?

**Solution.** $\angle C = 180° - (\angle CAB + \angle B) = 180° - (60° + 40°) = 80°$.
Let the bisector intersect $BC$ in $D$. $\angle CAD = \frac{1}{2} \angle CAB = 30°$.

$$\angle ADC = 180° - (\angle CAD + \angle C) = 180° - (30° + 80°) = 70°.$$
$$\angle ADB = 180° - \angle ADC = 180° - 70° = 110°.$$

**Ex. 84.** If a perpendicular is let fall from one end of the base of an isosceles triangle upon the opposite side, the angle formed by the perpendicular and the base equals one half of the vertical angle.



**Data:** Isos. $\triangle ABC$ and $AD \perp BC$.

**To prove** $\angle DAB = \frac{1}{2} \angle C$.

**Proof.** § 111, $\angle DAB = 90° - \angle B$,

and, Ax. 6, $2 \angle DAB = 180° - 2 \angle B$;

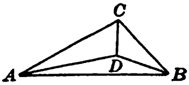
but, § 110, $C = 180° - (\angle A + \angle B) = 180° - 2 \angle B$;

$\therefore$ Ax. 1, $2 \angle DAB = \angle C$, or $\angle DAB = \frac{1}{2} \angle C$.

**Ex. 85.** If an angle of a triangle is equal to half the sum of the other two, what is the value of that angle?

**Solution.** For every degree in the required $\angle$ there are 2° in the sum of the other two and 3° in the sum of all the $\angle$s of the $\triangle$; hence the required $\angle = \frac{1}{3}$ of $180° = 60°$.

**Ex. 86.** The sum of the lines from a point within a triangle to the vertices of the triangle is less than the sum of the sides of the triangle and greater than half that sum.



**Data:** $\triangle ABC$ and the lines, $AD$, $BD$, and $CD$, from $D$, any point within to the vertices.

**To prove** 1. $AD + BD + CD < AB + BC + AC$.
2. $AD + BD + CD > \frac{1}{2}(AB + BC + AC)$.

**Proof.** 1. § 131, $AD + BD < AC + BC$,
$BD + CD < AB + AC$,
and $AD + CD < AB + BC$;

$\therefore$ adding, $2AD + 2BD + 2CD < 2AB + 2BC + 2AC$,

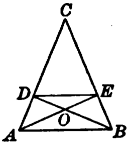
or $AD + BD + CD < AB + BC + AC$.

2. § 125, $AD + BD > AB$, $BD + CD > BC$, and $AD + CD > AC$;

$\therefore$ adding, $2AD + 2BD + 2CD > AB + BC + AC$,

or $AD + BD + CD > \frac{1}{2}(AB + BC + AC)$.

**Ex. 87.** *ABC* is an isosceles triangle having a vertical angle of 30°. From each extremity of the base perpendiculars are drawn to the opposite sides. What angles are formed at the intersection of these perpendiculars?



**Solution.** Let *AE* and *BD* be the $\perp$s intersecting at *O*. Then, § 111, $\angle CAE = 90° - \angle C = 90° - 30° = 60°$, and, § 115, $\angle EOD = \angle DAO + \angle ADO = 60° + 90° = 150°$. Also, § 55, $\angle AOD = 180° - \angle EOD = 180° - 150° = 30°$.

**Ex. 88.** The exterior angle at the vertex of an isosceles triangle is 110°. How many degrees are there in each angle of the triangle?

**Solution.** § 55, the $\angle$ at the vertex $= 180° - 110° = 70°$. § 115, the sum of the base $\angle$s $= 110°$, but they are equal; $\therefore$ each base $\angle = \frac{1}{2}$ of $110° = 55°$.

**Ex. 89.** The exterior angle at the base of an isosceles triangle is 110°. How many degrees are there in each angle of the triangle?

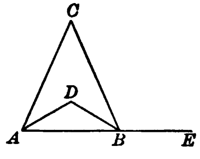
**Solution.** §§ 55, 116, each base $\angle = 180° - 110° = 70°$. Now, the sum of the base $\angle$s $= 70° \times 2 = 140°$, and the vert. $\angle = 180° - 140° = 40°$.

**Ex. 90.** The angle *C* at the vertex of the isosceles triangle *ABC* is one fourth of the exterior angle at *C*. How many degrees are there in angle *A*? In the exterior angle at *B*?

**Solution.** $\angle C = \frac{1}{4}$ of the ext. $\angle$ at *C*; then, $\angle C = \frac{1}{5}$ of the sum of the ext. $\angle$ and the int. $\angle$ at $C = \frac{1}{5}$ of $180° = 36°$. $\angle A + \angle B = 180° - \angle C = 144°$; but, § 116, $\angle A = \angle B$; $\therefore$ $\angle$s *A* and *B* each $= \frac{1}{2}$ of $144° = 72°$, and, § 55, the ext. $\angle$ at $B = 180° - 72° = 108°$.

**Ex. 91.** The angle formed by the bisectors of the base angles of an isosceles triangle is equal to an exterior angle at the base.

**Data**: Isos. $\triangle ABC$; the bisectors, *AD* and *BD*, of the base $\angle$s, meeting at *D*; and the ext. $\angle CBE$.



**To prove** $\angle ADB = \angle CBE$.

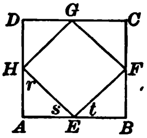
**Proof.** Ax. 9, $\angle ABD + \angle CBD = \angle ABC$;

but, Ax. 7, $\angle DAB = \angle CBD$;

$\therefore$ $\angle ABD + \angle DAB = \angle ABC$.

Now, § 110, $\angle ADB = 2$ rt. $\angle$s $- (\angle DAB + \angle ABD) = 2$ rt. $\angle$s $- \angle ABC$; but, § 55, $\angle CBE = 2$ rt. $\angle$s $- \angle ABC$; $\therefore$ Ax. 1, $\angle ADB = \angle CBE$.

**Ex. 92.** The lines joining in succession the middle points of the sides of a square form a square.



**Data**: Square $ABCD$, and the lines $EF$, $FG$, $GH$, and $HE$ joining the middle points of its sides in succession.

**To prove** $EFGH$ a square.

**Proof.** Since, Ax. 7, $AH = AE = BE = BF =$ etc.,

and, § 52, $\angle A = \angle B =$ etc., § 100, $\triangle AEH = \triangle BEF =$ etc.,

and $HE = EF = FG = GH$.

§ 116, $\angle r = \angle s$, and, § 111, each $\angle = \frac{1}{2}$ rt. $\angle$;

similarly, $\angle t = \frac{1}{2}$ rt. $\angle$.

Hence, § 56, $\angle HEF = 2$ rt. $\angle$s $- (\angle s + \angle t) = 1$ rt. $\angle$.

Similarly, $\angle$s $EFG$, $FGH$, and $GHE$ are rt. $\angle$s.

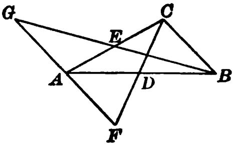
§ 26, $EH \perp EF$, and $FG \perp EF$; $\therefore$ § 71, $EH \parallel FG$; similarly, $EF \parallel HG$, and $EFGH$ is a ▱.

Hence, § 143, $EFGH$ is a square.

**Ex. 93.** The sum of any two angles of a parallelogram, which are not opposite, equals two right angles, and the sum of the angles of a parallelogram equals four right angles.

**Proof.** Any two $\angle$s of a ▱, which are not opposite, are the int. $\angle$s on the same side of a transversal cutting two parallel lines; hence, § 78, their sum $= 2$ rt. $\angle$s. Similarly, the sum of the other two $\angle$s of the ▱ $= 2$ rt. $\angle$s. Consequently, the sum of the $\angle$s of a ▱ $= 2$ rt. $\angle$s $+ 2$ rt. $\angle$s $= 4$ rt. $\angle$s.

**Ex. 94.** If medians are drawn from two vertices of a triangle and each is produced its own length, the line joining the extremities of the produced medians and the other vertex of the triangle is a straight line.



**Data**: $\triangle ABC$, and any two medians, $BE$ and $CD$, each produced its own length to $G$ and $F$ respectively; also line $FAG$.

**To prove** $FAG$ a straight line.

**Proof.** Data, $GE = BE$, $AE = EC$, and § 59, $\angle AEG = \angle BEC$; $\therefore$ § 100, $\triangle AEG = \triangle BEC$,

and $\angle ECB = \angle EAG$; similarly, $\angle FAD = \angle ABC$;

but, § 110, $\angle ACB + \angle BAC + \angle ABC = 2$ rt. $\angle$s;

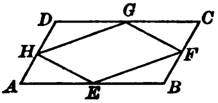
$\therefore$ substituting, $\angle$s $EAG$ and $FAD$ for their equals,

$$\angle EAG + \angle BAC + \angle FAD = 2 \text{ rt. } \angle\text{s},$$

or, Ax. 9, $\angle BAG + \angle FAD = 2$ rt $\angle$s.;

hence, § 58, $FAG$ is a straight line.

**Ex. 95.** If the sides of a parallelogram are bisected and these middle points are joined in succession, the connecting lines form a parallelogram.



**Data:** ▱ $ABCD$, and the lines $EF$, $FG$, $GH$, and $HE$ joining the middle points of its sides in succession.

**To prove** $EFGH$ a parallelogram.

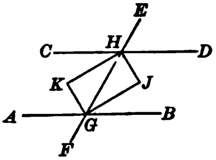
**Proof.** §§ 140, 151, $BC = AD$, and $AB = DC$;

$\therefore$ Ax. 7, $BF = HD$, and $BE = DG$; also, § 149, $\angle EBF = \angle GDH$;

$\therefore$ § 100, $\triangle EBF = \triangle GDH$, and $EF = HG$.

Similarly, $HE = FG$; hence, § 148, $EFGH$ is a parallelogram.

**Ex. 96.** If the four interior angles formed by a transversal crossing two parallel lines are bisected and the bisectors produced until they meet, the figure thus formed is a rectangle.



**Data:** Parallel lines $AB$ and $CD$ cut by the transversal $EF$ in $G$ and $H$, respectively; and the bisectors of the four int. ∠s produced to form the figure $GJHK$.

**To prove** $GJHK$ a rectangle.

**Proof.** Ex. 25, $KH \parallel GJ$ and $KG \parallel HJ$; $\therefore$ § 140, $GJHK$ is a ▱.

Ex. 14, $\angle KHJ = 90° =$ a rt. $\angle$; similarly $\angle KGJ$ is a rt. $\angle$.

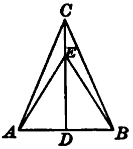
$\angle DHG + \angle BGH =$ 2 rt. ∠s; $\therefore$ Ax. 7, $\angle JHG + \angle JGH =$ 1 rt. $\angle$;

then, § 110, $\angle J =$ 2 rt. ∠s $- (\angle JHG + \angle JGH) =$ 1 rt. $\angle$;

hence, § 149, $\angle K$ is a rt. $\angle$.

Therefore, § 141, $GJHK$ is a rectangle.

**Ex. 97.** If a line is drawn through the vertices of two isosceles triangles on the same base it bisects the base.

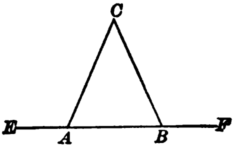


**Data:** Two isos. △s $ABC$ and $ABE$ on the same base $AB$, and the line through $C$ and $E$ meeting $AB$ in $D$.

**To prove** $CD$ bisects $AB$.

**Proof.** § 90, $C$ and $E$ are each equidistant from $A$ and $B$; $\therefore$ § 106, $CD \perp AB$ at its middle point; that is, $CD$ bisects $AB$.

**Ex. 98.** If two equal straight lines are drawn from a point to a line, the angles which they form with the given line are equal.

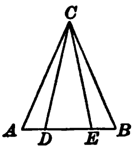


**Data:** Two equal straight lines, $CA$ and $CB$, drawn from the point $C$, meeting the line $EF$ in $A$ and $B$ respectively, and forming ∠s $BAC$ and $ABC$.

**To prove** $\angle BAC = \angle ABC$.

**Proof.** Data, $AC = BC$; $\therefore$ § 90, $\triangle ABC$ is isos. and, § 116, $\angle BAC = \angle ABC$.

**Ex. 99.** If lines are drawn from the vertex of an isosceles triangle to points in the base equally distant from its extremities, they are equal.



**Data**: Isos. $\triangle ABC$ and lines $CD$ and $CE$ drawn from $C$ and meeting $AB$ in points $D$ and $E$ equally distant from $A$ and $B$ respectively.

**To prove** $CD = CE.$

**Proof.** Data, $AD = BE$, § 90, $AC = BC$, and, § 116, $\angle CAD = \angle CBE$; $\therefore \triangle ADC = \triangle BEC$, and $CD = CE$.

PROP. XXXVIII, page 63.

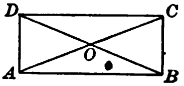
**Proof.** In $\triangle$s $ABC$ and $ADC$, $AC$ is common,

§ 73, $\angle r = \angle t$, and $\angle s = \angle v$;

$\therefore$ § 102, $\triangle ABC = \triangle ADC.$

**Ex. 100.** If the diagonals of a quadrilateral are equal and bisect each other, the figure is a rectangle.

**Data**: A quadrilateral $ABCD$ whose diagonals $AC$ and $BD$ bisect each other at $O$.



**To prove** $ABCD$ a rectangle.

**Proof.** Data, $AO = CO$, $BO = DO$, and, § 59, $\angle AOB = \angle DOC$; $\therefore$ § 100, $\triangle ABO = \triangle DCO$, and $AB = DC$, also $\angle OAB = \angle OCD$; $\therefore AB \parallel DC$; hence, § 150, $ABCD$ is a ▱.

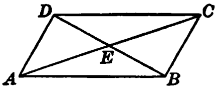
Now, data, $CO = BO = \frac{1}{2} BD$; $\therefore$ Ex. 64, $\angle BCD$ is a rt. $\angle$; similarly, each $\angle$ of $ABCD$ is a rt. $\angle$; hence, § 141, $ABCD$ is a rectangle.

**Ex. 101.** From a figure representing a parallelogram and its diagonals, select four pairs of equal triangles.

(1) $\triangle$s $ABE$ and $DCE$; (2) $\triangle$s $AED$ and $BEC$;

(3) $\triangle$s $ACD$ and $ABC$; (4) $\triangle$s $ABD$ and $BCD$.

**Proof.** (1) § 154, $AE = CE$, $BE = DE$, and, § 59, $\angle AEB = \angle DEC$; $\therefore \triangle ABE = \triangle DCE$.



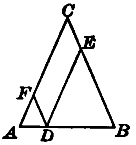
(2) Similarly, $\triangle AED = \triangle BEC.$

(3) § 152, $\triangle ACD = \triangle ABC.$

(4) § 152, $\triangle ABD = \triangle BCD.$

**Ex. 102.** If from any point in the base of an isosceles triangle lines are drawn parallel to the equal sides and produced until they meet the sides of the triangle, the sum of these lines is equal to one of the equal sides of the triangle.

**Data**: Isos. $\triangle ABC$; $D$ any point in the base; and lines $DF$ and $DE$ parallel to $BC$ and $AC$ respectively, and meeting $AC$ and $BC$ in $F$ and $E$ respectively.



**To prove** $DF + DE = AC.$

**Proof.** § 76, $\angle ADF = \angle B$; § 116, $\angle A = \angle B$;

$\therefore \angle A = \angle ADF$, and, § 118, $DF = AF$.

**Also**, since, § 140, $FDEC$ is a parallelogram, § 153, $DE = FC$.
Hence, Ax. 2, $DF + DE = AF + FC$, or $DF + DE = AC$.

Ex. **103**. In the triangle $ABC$ angle $A$ is double angle $B$ and the exterior angle at $C$ is 105°. How many degrees are there in angles $A$ and $B$ respectively?

**Solution.** Since the ext. $\angle$ at $C$ is $105°$, § 115, $\angle A + \angle B = 105°$. For every degree in $\angle B$ there are $2°$ in $\angle A$, and $3°$ in the sum of $\angle$s $A$ and $B$; therefore, $\angle B = \frac{1}{3}$ of $105° = 35°$, and $\angle A = 105° - 35° = 70°$.

Ex. **104**. If one angle of a parallelogram is a right angle, each of the other angles is a right angle.

**Proof.** § 149, the $\angle$ opposite the given $\angle$ is a rt. $\angle$.

Then, these two opp. $\angle$s $= 2$ rt. $\angle$s; but, Ex. 93, the sum of all the $\angle$s $= 4$ rt. $\angle$s; $\therefore$ the sum of the other two opp. $\angle$s $= 2$ rt. $\angle$s, and since they are equal, each one is a rt. $\angle$.

Ex. **105**. One angle of a parallelogram is three times its supplement. What is the value of each angle of the parallelogram?

**Solution.** Suppose $\angle A$ to be the given $\angle$ of the ▱ $ABCD$.

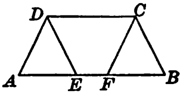
Then, §§ 78, 32, $\angle B$ is sup. of $\angle A$;

$$\therefore \angle A = 3\angle B, \text{ and } \angle A + \angle B = 4\angle B = 180°;$$

$$\therefore \angle B = \tfrac{1}{4} \text{ of } 180° = 45°, \text{ and } \angle A = 3 \times 45° = 135°;$$

consequently, § 149, $\angle C$ (opp. $\angle A$) $= 135°$, and $\angle D$ (opp. $\angle B$) $= 45°$.

Ex. **106**. If from the extremities of the shorter base of an isosceles trapezoid lines are drawn parallel to the equal sides, two equal triangles are formed.

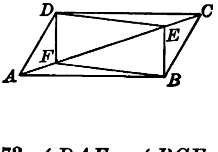


**Data:** Isos. trap. $ABCD$, and lines from $D$ and $C$ parallel to $CB$ and $DA$ respectively and meeting $AB$ in $E$ and $F$ respectively.

**To prove** $\triangle AED = \triangle FBC$.

**Proof.** § 140, $AFCD$ and $EBCD$ are ▱s; $\therefore$ § 153, $AD = FC$, $ED = BC$, and, § 81, $\angle ADE = \angle FCB$; hence, § 100, $\triangle AED = \triangle FBC$.

Ex. **107**. If in a parallelogram any two points in a diagonal equally distant from its extremities are joined to the vertices of the opposite angles, the figure thus formed is a parallelogram.



**Data:** $F$ and $E$, two points in the diagonal $AC$ of the ▱ $ABCD$ equally distant from its extremities and the lines joining $F$ and $E$ with the vertices $D$ and $B$.

**To prove** $FBED$ a parallelogram.

**Proof.** Data, $AF = EC$, § 153, $AD = BC$, and, § 73, $\angle DAF = \angle BCE$; $\therefore$ § 100, $\triangle AFD = \triangle BCE$, and $FD = BE$; similarly, $FB = DE$; hence, § 148, $FBED$ is a parallelogram.

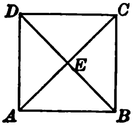
Ex. **108**. How many degrees are there in each angle of an equiangular polygon of five sides?

**Solution.** § 166, the sum of the $\angle$s $= (n - 2)2$ rt. $\angle$s $= (5 - 2)2$ rt. $\angle$s $= 6$ rt. $\angle$s $= 540°$, and since there are 5 $\angle$s each one $= \frac{1}{5}$ of $540° = 108°$.

Ex. **109**. How many sides has a polygon the sum of whose interior angles is double the sum of its exterior angles?

**Solution.** § 167, the sum of the ext. $\angle$s $= 4$ rt. $\angle$s; $\therefore$ the sum of the int. $\angle$s $= 2 \times 4$ rt. $\angle$s $= 8$ rt. $\angle$s; but, § 166, the sum of the int. $\angle$s $= (n - 2) \times 2$ rt. $\angle$s; $\therefore$ $(n - 2)2$ rt. $\angle$s $= 8$ rt. $\angle$s, or $(n - 2)2 = 8$; whence, $n = 6$.

Ex. **110**. The angle formed by the diagonals of a square is a right angle.

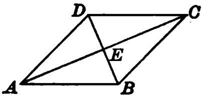


**Data**: The square $ABCD$ whose diagonals $AC$ and $BD$ intersect at $E$.

**To prove** $\angle AEB$ a rt. $\angle$.

**Proof.** § 143, $D$ and $B$ are each equidistant from $A$ and $C$; $\therefore$ § 106, $BD \perp AC$; hence, § 26, $\angle AEB$ is a rt. $\angle$.

Ex. **111**. The diagonals of a rhombus bisect each other at right angles.

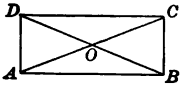


**Data**: The rhombus $ABCD$ whose diagonals are $AC$ and $BD$.

**To prove** $AC$ and $BD$ bisect each other at rt. $\angle$s.

**Proof.** § 144, $B$ and $D$ are each equidistant from $A$ and $C$; $\therefore$ § 106, $BD \perp AC$ at its middle pt.; similarly, $AC \perp BD$ at its middle pt.; that is, $AC$ and $BD$ bisect each other at rt. $\angle$s.

Ex. **112**. The diagonals of a rectangle are equal.

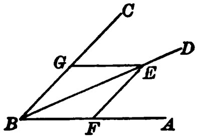


**Data**: Rectangle $ABCD$, and its diagonals $AC$ and $BD$.

**To prove** $AC = BD$.

**Proof.** In $\triangle$s $ABC$ and $ABD$, $AB$ is common, § 153, $BC = AD$, and, §§ 141, 52, $\angle ABC = \angle BAD$; $\therefore$ § 100, $\triangle ABC = \triangle ABD$, and $AC = BD$.

Ex. **113**. If from any point in the bisector of an angle straight lines are drawn parallel to the sides of the angle and produced to meet the sides, the figure thus formed is a parallelogram.



**Data**: $\angle ABC$ and its bisector $BD$; also $E$, any point in $BD$, and the lines $EG$ and $EF$ drawn parallel to $AB$ and $CB$ respectively, meeting $CB$ and $AB$ in $G$ and $F$ respectively.

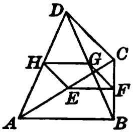
**To prove** $BFEG$ a parallelogram.

**Proof.** Data, $EG \parallel FB$, and $EF \parallel GB$; $\therefore$ § 140, $BEFG$ is a parallelogram.

Ex. **114**. The difference between two angles of a parallelogram which have a common side is 60°. What is the value of each angle of the parallelogram?

**Solution.** § 78, the sum of the two $\angle s = 2$ rt. $\angle s = 180°$; but, data, the difference between the $\angle s = 60°$; hence, $\frac{1}{2}(180° + 60°) = 120°$, the greater of the two $\angle s$, and $180° - 120° = 60°$, the less of the two $\angle s$. § 149, the other $\angle s$ of the $\square$ are respectively 120° and 60°.

Ex. **115**. If the middle points of any two opposite sides of a quadrilateral are joined to each of the middle points of the diagonals, the four joining lines will form a parallelogram.

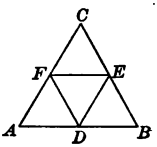


**Data**: Quadrilateral $ABCD$; $H$ and $F$, the middle points of sides $AD$ and $BC$; and $E$ and $G$, the middle points of the diagonals; also the lines $HE$, $HG$, $FE$, and $FG$.

**To prove** $EFGH$ a parallelogram.

**Proof.** In $\triangle ABC$, § 159, $EF \parallel AB$, and in $\triangle ABD$, $HG \parallel AB$; $\therefore$ § 80, $EF \parallel HG$; similarly, $EH \parallel FG$; hence, § 140, $EFGH$ is a parallelogram.

Ex. **116**. If the middle points of the sides of an equilateral triangle are joined, the triangles formed are equilateral.

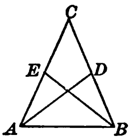


**Data**: Equilateral $\triangle ABC$, and the lines $DE$, $EF$, and $DF$ joining the middle points of its sides.

**To prove** $\triangle s$ $ADF$, $DBE$, etc., equilateral.

**Proof.** Data, and, Ax. 7, $AD = DB = BE = CE = FC = AF$, also, § 158, $FE = \frac{1}{2}AB = AD$; similarly, $DF = BE$, and $DE = AF$; $\therefore$ Ax. 1, $AD = DB = BE = CE = FC = AF = FE = DF = DE$, and the triangles are equilateral.

Ex. **117**. The lines drawn from the middle points of the equal sides of an isosceles triangle to the opposite extremities of the base are equal.



**Data**: Isos. $\triangle ABC$, and the medians $AD$ and $BE$ from the extremities of the base.

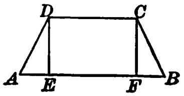
**To prove** $AD = BE$.

**Proof.** In $\triangle s$ $ABD$ and $ABE$, $AB$ is common, Ax. 7, $BD = AE$, and, § 116, $\angle ABD = \angle BAE$; $\therefore$ $\triangle ABD = \triangle ABE$, and $AD = BE$.

Ex. **118**. The parallel sides of a trapezoid are 35 and 55 feet respectively. What is the length of the line joining the middle points of the non-parallel sides?

**Solution.** § 160, the required line $= \frac{1}{2}(35 \text{ ft.} + 55 \text{ ft.}) = 45$ ft.

Ex. **119.** If from the extremities of the shorter base of an isosceles trapezoid perpendiculars are drawn to the longer base, two equal triangles are formed.



**Data:** Isos. trap. $ABCD$ and perpendiculars $DE$ and $CF$ from $D$ and $C$ respectively to $AB$.

**To prove** $\triangle AED = \triangle FBC$.

**Proof.** Data, $DE \perp AB$, and $CF \perp AB$; $\therefore$ § 71, $DE \parallel CF$; hence, § 151, $DE = CF$. Also, § 139, $AD = BC$; $\therefore$ § 123, $\triangle AED = \triangle FBC$.

Ex. **120.** How many sides has a polygon the sum of whose exterior angles is double the sum of its interior angles?

**Solution.** Since, § 167, the sum of the ext. $\angle$s $= 4$ rt. $\angle$s, data, the sum of the int. $\angle$s is 2 rt. $\angle$s.

Then, § 166, $(n-2)\,2$ rt. $\angle$s $= 2$ rt. $\angle$s; whence, $2n - 4 = 2$, and $n = 3$.

Ex. **121.** How many sides has a polygon the sum of whose interior angles is equal to the sum of its exterior angles?

**Solution.** Since, § 167, the sum of the ext. $\angle$s $= 4$ rt. $\angle$s, the sum of the int. $\angle$s $= 4$ rt. $\angle$s.

Then, § 166, $(n-2)\,2$ rt. $\angle$s $= 4$ rt. $\angle$s; whence, $2n - 4 = 4$, and $n = 4$.

Ex. **122.** The perimeter of an isosceles triangle is 176 feet and the base $1\frac{1}{5}$ times one of the equal sides. What is the length of each side of the triangle?

**Solution.** Denote each equal side by $x$.

Then, the base $= \frac{6}{5}x$ and the perimeter $= \frac{6x}{5} + x + x = 176$ ft.; whence, $x = 55$ ft., and $\frac{6x}{5} = 66$ ft.

Hence, each of the equal sides is 55 ft., and the base is 66 ft.

Ex. **123.** How many sides has an equiangular polygon which can be divided into equilateral triangles by lines drawn from a point within to the vertices of the polygon?

**Solution.** Ex. 74, each $\angle$ of each $\triangle$ will contain 60°.

Now the sum of the $\angle$s about the point within is 360°, and there will be as many angles about that point as 60 is contained times in 360, which is 6; consequently there will be 6 triangles, and their exterior sides are the sides of the polygon; hence, the polygon has 6 sides.

Ex. **124.** If through a point halfway between two parallel lines two transversals are drawn, they intercept equal parts on the parallel lines.

**Data:** (See second Fig. on p. 78, Geom.) Two parallel lines, $AB$ and $CD$; the point $E$ halfway between them, and any two transversals through $E$, as $FG$ and $HJ$ intercepting the parts $FH$ and $JG$ on $AB$ and $CD$ respectively.

**To prove** $FH = JG$.

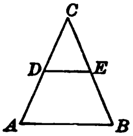
**Proof.** Draw $LK$ through $E$ and perpendicular to $CD$.

Then, § 72, $LK \perp AB$ and measures the distance between $AB$ and $CD$; consequently, $LE = KE$; also, § 73, $\angle EFL = \angle EGK$;

$\therefore$ § 114, $\triangle FEL = \triangle GEK$, and $FE = GE$.

Then, in $\triangle$s $FEH$ and $GEJ$, $FE = GE$, $\angle EFH = \angle EGJ$, and, § 59, $\angle FEH = \angle GEJ$; $\therefore$ § 102, $\triangle FEH = \triangle GEJ$, and $FH = JG$.

**Ex. 125.** A straight line cutting the sides of an isosceles triangle, and parallel to the base, makes equal angles with the sides.

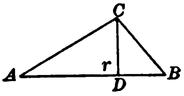


**Data**: Isos. $\triangle ABC$, and line $DE \parallel AB$ and cutting the sides $AC$ and $BC$ in $D$ and $E$, respectively.

**To prove** $\angle CDE = \angle CED$.

**Proof.** § 76, $\angle CDE = \angle A$, and $\angle CED = \angle B$; but, § 116, $\angle A = \angle B$; $\therefore \angle CDE = \angle CED$.

Ex. **126**. If the base of a triangle is divided into two parts by a perpendicular from the vertex, each part of the base is less than the adjacent side of the triangle.



**Data**: $\triangle ABC$, and $CD \perp AB$, dividing $AB$ into the parts $AD$ and $BD$.

**To prove** $AD < AC$, and $BD < BC$.

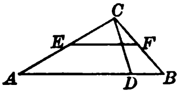
**Proof.** Since, § 26, $\angle r$ is a rt. $\angle$, $\triangle ADC$ is a rt. $\triangle$, and $AC$ is its hypotenuse.

Hence, § 128, $AC > AD$; that is, $AD < AC$.

Similarly, it may be proved that $BD < BC$.

**Ex. 127**. Any straight line drawn from the vertex of a triangle to the base is bisected by the straight line which joins the middle points of the other sides of the triangle.

**Data**: $\triangle ABC$; any line, as $CD$, from the vertex to the base $AB$; and $EF$ joining the middle points of $AC$ and $BC$ and intersecting $CD$.

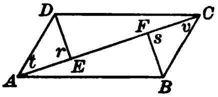


**To prove** $EF$ bisects $CD$.

**Proof.** Data, § 159, $EF \parallel AB$.

Then, in $\triangle ADC$, since, data, $EF$ bisects $AC$, § 158, $EF$ bisects $CD$.

**Ex. 128.** The perpendiculars to the diagonal of a parallelogram from the opposite vertices are equal.



**Data**: $\square ABCD$; a diagonal $AC$; and the $\perp$s $DE$ and $BF$, to $AC$ from the opposite vertices.

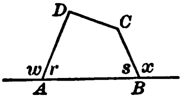
**To prove** $DE = BF$.

**Proof.** Data, § 26, $\angle$s $r$ and $s$ are rt. $\angle$s.

Then, in rt. $\triangle$s $AED$ and $BFC$, § 153, $AD = BC$, and, § 73, $\angle t = \angle v$; $\therefore$ § 114, $\triangle AED = \triangle BFC$, and $DE = BF$.

**Ex. 129.** If one side of a quadrilateral is extended in both directions, the sum of the exterior angles formed is equal to the sum of the two interior angles opposite the side produced.

**Data:** Quadrilateral $ABCD$, whose side $AB$ is extended in both directions, forming the ext. $\angle$s $w$ and $x$.



**To prove** $\angle w + \angle x = \angle C + \angle D.$

**Proof.** § 55, $\angle w + \angle r = 2$ rt. $\angle$s,

and $\angle s + \angle x = 2$ rt. $\angle$s;

$\therefore$ Ax. 2, $\angle w + \angle r + \angle s + \angle x = 4$ rt. $\angle$s;

but, § 166, $\angle r + \angle s + \angle C + \angle D = 4$ rt. $\angle$s:

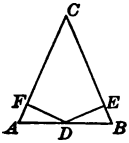
$\therefore$ Ax. 1, $\angle w + \angle r + \angle s + \angle x = \angle r + \angle s + \angle C + \angle D.$

Taking $\angle r + \angle s$ from each member of this equation,

Ax. 3, $\angle w + \angle x = \angle C + \angle D.$

**Ex. 130.** If in an isosceles triangle perpendiculars are drawn from the middle point of the base to the equal sides, the perpendiculars are equal.

**Data:** Isos. $\triangle ABC$; $D$ the middle point of the base; and $\perp$s $DF$ and $DE$ to $AC$ and $BC$, respectively.



**To prove** $DF = DE.$

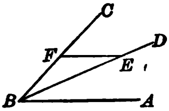
**Proof.** In rt. $\triangle$s $ADF$ and $BDE$, data, $AD = BD$,

and, § 116, $\angle DAF = \angle DBE$;

$\therefore$ § 114, $\triangle ADF = \triangle BDE$, and $DF = DE$.

**Ex. 131.** A straight line drawn from any point in the bisector of an angle to either side and parallel to the other side makes, with the bisector and the side it meets, an isosceles triangle.

**Data:** $\angle ABC$; its bisector $BD$; $E$ any point in $BD$; and $EF \parallel AB$ meeting $BC$ in $F$ and forming $\triangle BEF$.



**To prove** $\triangle BEF$ isosceles.

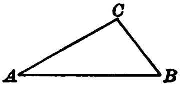
**Proof.** Data, $\angle FBE = \angle ABE$,

and, § 73, $\angle FEB = \angle ABE$;

$\therefore$ Ax. 1, $\angle FBE = \angle FEB$; hence, § 118, $\triangle BEF$ is isosceles.

**Ex. 132.** The difference between two sides of a triangle is less than the third side.

**Data:** Any $\triangle$, as $ABC$, and any two sides, as $AB$ and $AC$ of which $AC$ is the less.

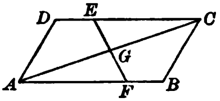


**To prove** $AB - AC < BC.$

**Proof.** § 124, $AB < AC + BC.$

Taking $AC$ from each side of this inequality, Ax. 5, $AB - AC < BC.$

**Ex. 133.** Any straight line through the middle point of a diagonal of a parallelogram, and terminated by the opposite sides, is bisected at that point.

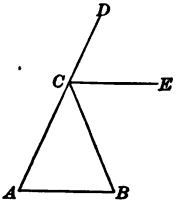


**Data:** $\square ABCD$, and any line, as $EF$, through $G$, the middle point of diagonal $AC$, terminated in $DC$ and $AB$.

**To prove** $FG = EG$.

**Proof.** Data, $AG = GC$, § 59, $\angle AGF = \angle CGE$, and, § 73, $\angle GAF = \angle GCE$; $\therefore$ § 102, $\triangle AFG = \triangle EGC$, and $FG = EG$.

Ex. **134.** If either of the equal sides of an isosceles triangle is produced through the vertex, the line bisecting the exterior angle thus formed is parallel to the base of the triangle.



**Data:** Isos. $\triangle ABC$, and $CE$ bisecting the ext. $\angle BCD$ formed by producing $AC$, one of the equal sides.

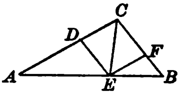
**To prove** $CE \parallel AB$.

**Proof.** §§ 115, 116, $\angle BCD = \angle A + \angle B = 2\angle B$;

$\therefore \angle BCE = \frac{1}{2}\angle BCD = \angle B$.

Hence, § 75, $CE \parallel AB$.

Ex. **135.** If the bisector of one of the angles of a triangle meets the opposite side, the lines from the point of meeting, parallel to the other sides and terminated by them, are equal.

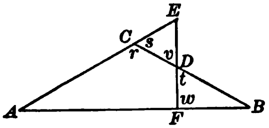


**Data:** $\triangle ABC$; the bisector $CE$ of one of its $\angle s$, $C$, meeting the opposite side $AB$ in $E$; also $EF$ and $ED$ parallel to $AC$ and $BC$, resp., and meeting $BC$ and $AC$ in $F$ and $D$, resp.

**To prove** $EF = ED$.

**Proof.** Data, § 140, $CDEF$ is a $\square$, and $CE$ is its diagonal; $\therefore$ § 152, $\triangle FCE = \triangle DCE$, and since, data, $\angle FCE = \angle DCE$, § 108, $EF = DE$.

Ex. **136.** If each of the angles at the base of an isosceles triangle is one fourth the vertical angle, every line perpendicular to the base (except the one that passes through the vertex) forms an equilateral triangle with the other two sides, produced when necessary.



**Data:** Isos. $\triangle ABC$, whose base $\angle s$ $A$ and $B$ are each one fourth vert. $\angle C$; also any line, as $FE$, $\perp AB$ and intersecting the sides $BC$ and $AE$ (produced if necessary) in $D$ and $E$.

**To prove** $\triangle$ $CDE$ equilateral.

**Proof.** Data, $\angle A = \frac{1}{4}\angle r$; $\therefore 4\angle A = \angle r$, and since, § 116, $\angle A = \angle B$,

$$\angle A + \angle B + \angle r = \angle A + \angle A + 4\angle A = 6\angle A;$$

but, § 110, $\angle A + \angle B + \angle r = 180°;$

$$\therefore 6\angle A = 180°, \text{ and } \angle A = 30°.$$

Hence, $\angle r = 180° - (\angle A + \angle B) = 180° - (30° + 30°) = 120°;$

$\therefore$ § 55, $\angle s = 180° - \angle r = 180° - 120° = 60°.$

Now, $\angle t = 180° - (\angle B + \angle w) = 180° - (30° + 90°) = 60°;$

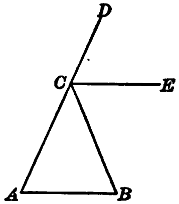
$\therefore$ § 59, $\angle v = \angle t = 60°,$

and $\angle E = 180° - (\angle s + \angle v) = 180° - (60° + 60°) = 60°.$

Hence, $\triangle CDE$ is equiangular;

$\therefore$ § 119, $\triangle CDE$ is equilateral.

Ex. **137**. If the straight line bisecting an exterior angle of a triangle is parallel to a side, the triangle is isosceles.



**Data:** $\triangle ABC$ such that the straight line $CE$, bisecting the ext. $\angle$ at $C$, is parallel to $AB$.

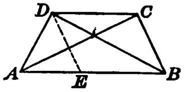
**To prove** $\triangle ABC$ isosceles.

**Proof.** § 76, $\angle A = \angle ECD$, § 73, $\angle B = \angle BCE$;

but, data, $\angle ECD = \angle BCE$;

$\therefore \angle A = \angle B$; hence, § 118, $\triangle ABC$ is isosceles.

Ex. **138**. If the non-parallel sides of a trapezoid are equal, the base angles are equal, and the diagonals are equal.



**Data:** Trapezoid $ABCD$, having its non-parallel sides, $AD$ and $BC$, equal; also its diagonals $AC$ and $BD$.

**To prove** 1. $\angle DAB = \angle ABC$.

2. $AC = BD$.

**Proof.** 1. From $D$ draw $DE \parallel BC$.

Then, § 151, $ED = BC$; but, data, $AD = BC$; $\therefore$ Ax. 1, $ED = AD$;

that is, § 90, $\triangle AED$ is isos., and, § 116, $\angle DAE = \angle AED$;

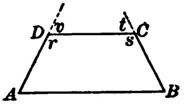
§ 76, $\angle ABC = \angle AED$; $\therefore$ Ax. 1, $\angle DAE = \angle ABC$.

2. In $\triangle$s $ABC$ and $ABD$, $AB$ is common, $AD = BC$,

and, from 1, $\angle ABC = \angle DAB$; $\therefore$ § 100, $\triangle ABC = \triangle ABD$, and $AC = BD$.

**Cor.** *If the base angles of a trapezoid are equal the trapezoid is isosceles.*

**Ex. 139.** If the angles adjacent to one base of a trapezoid are equal, those adjacent to the other base are also equal.



**Data**: Trapezoid $ABCD$, in which $\angle A = \angle B$.

**To prove** $\angle r = \angle s$.

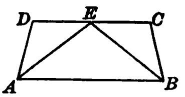
**Proof.** Produce $AD$ and $BC$ through $D$ and $C$, respectively, forming $\angle v$ and $\angle t$.

§ 76, $\angle v = \angle A$, and $\angle t = \angle B$; but, data, $\angle A = \angle B$; $\therefore\ \angle v = \angle t$;

Now, §§ 55, 32, $\angle r$ is supplement of $\angle v$, and $\angle s$ is supplement of $\angle t$;

$\therefore$ § 54, $\angle r = \angle s$.

**Ex. 140.** If the upper base of an isosceles trapezoid equals the sum of the non-parallel sides, lines from the middle point of the upper base to the extremities of the lower divide the figure into three isosceles triangles.



**Data**: Isos. trap. $ABCD$ in which $DC = AD + BC$; $E$ the middle pt. of $DC$, and the lines $AE$ and $BE$.

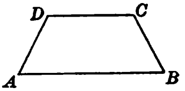
**To prove** $\triangle$s $AED$, $BCE$, and $ABE$ isosceles.

**Proof.** $AD + BC = DE + EC$; but, § 139, $AD = BC$, and, data, $DE = EC$; $\therefore\ 2\,AD = 2\,DE$, or $AD = DE$, and, § 90, $\triangle AED$ is isosceles.

Similarly, $BC = EC$, and $\triangle BCE$ is isosceles.

Ex. 138, $\angle DAB = \angle ABC$; $\therefore$ Ex. 139, $\angle D = \angle C$, and since $AD = BC$, and $DE = EC$, § 100, $\triangle AED = \triangle BCE$, and $AE = BE$; $\therefore\ \triangle ABE$ is isosceles.

**Ex. 141.** The opposite angles of an isosceles trapezoid are supplements of each other.



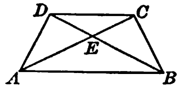
**Datum**: Isos. trap. $ABCD$.

**To prove** $\angle A$ the supplement of $\angle C$, and $\angle B$ the supplement of $\angle D$.

**Proof.** §§ 78, 32, $\angle B$ is sup. of $\angle C$; but, Ex. 138, $\angle A = \angle B$; $\therefore\ \angle A$ is sup. $\angle C$.

Similarly, $\angle B$ is supplement of $\angle D$.

**Ex. 142.** The segments of the diagonals of an isosceles trapezoid form with the upper and lower bases two isosceles triangles.



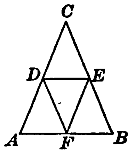
**Data**: Isos. trap. $ABCD$, and its diagonals $AC$ and $BD$ intersecting at $E$.

**To prove** $\triangle$s $ABE$ and $CDE$ isosceles.

**Proof.** In $\triangle$s $ABC$ and $ABD$, $AB$ is common, § 139, $BC = AD$, and Ex. 138, $\angle ABC = \angle BAD$; $\therefore$ § 100, $\triangle ABC = \triangle ABD$, and $\angle BAC = \angle ABD$; that is, in $\triangle ABE$, $\angle BAE = \angle ABE$; $\therefore$ § 118, $\triangle ABE$ is isosceles.

Similarly, $\triangle CDE$ is isosceles.

Ex. **143**. The triangle formed by joining the middle points of the sides of an isosceles triangle is isosceles.

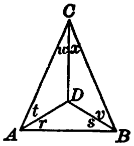


**Data**: Isos. $\triangle ABC$ in which $AC = BC$; and the lines $FE$, $ED$, and $DF$ joining the middle points of its sides, forming $\triangle DEF$.

**To prove** $\triangle DEF$ isosceles.

**Proof.** § 158, $FE = \frac{1}{2}AC$, and $FD = \frac{1}{2}BC$; but, data, $AC = BC$; $\therefore$ Ax. 7, $FE = FD$, and, § 90, $\triangle DEF$ is isosceles.

Ex. **144**. If the two angles at the base of an isosceles triangle are bisected, the line which joins the intersection of the bisectors with the vertex of the triangle bisects the vertical angle.



**Data**: Isos. $\triangle ABC$; the bisectors, $AD$ and $BD$, of its base angles, intersecting at $D$; and the line $CD$.

**To prove** $CD$ bisects $\angle ACB$.

**Proof.** § 116, $\angle CAB = \angle ABC$; $\therefore$ Ax. 7, $\angle r = \angle s$; hence, § 118, $AD = BD$.

In △s $ADC$ and $BDC$, $AD = BD$, data, $AC = BC$, and, Ax. 7, $\angle t = \angle v$; $\therefore \triangle ADC = \triangle BDC$, and $\angle w = \angle x$; that is, $CD$ bisects $\angle ACB$.

Or, since, § 169, the bisectors of the ∠s of a △ meet in a point, Ax. 12, the line $CD$, which joins the vertex $C$ with $D$ the intersection of the given bisectors, bisects $\angle ACB$.

Ex. **145**. $ABCD$ is a parallelogram; $E$ and $F$ are the middle points of $AD$ and $BC$ respectively. Show that $BE$ and $FD$ trisect the diagonal $AC$.

**Proof.** Let $BE$ and $FD$ cut $AC$ at $G$ and $H$, respectively.

§ 140, $AD \parallel BC$, § 153, $AD = BC$; $\therefore$ Ax. 7 and data, $ED = BF$; hence, § 150, $EBFD$ is a ▱, and, § 140, $EB \parallel DF$.

Then, in $\triangle GBC$, § 158, $GH = HC$, and in $\triangle ADH$, $AG = GH$;

$\therefore$ Ax. 1, $AG = GH = HC$; that is, $BE$ and $FD$ trisect $AC$.

Ex. **146**. The exterior angle of an equiangular hexagon is equal to the interior angle of an equiangular triangle.

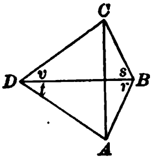
**Proof.** A hexagon has 6 ext. ∠s, and, § 167, their sum = 4 rt. ∠s.

Now, since the hexagon is equiangular, § 54, its ext. ∠s are all equal, and each $\angle = \frac{1}{6}$ of 4 rt. ∠s $= \frac{2}{3}$ rt. ∠.

But, each ∠ of an equiangular △ $= \frac{1}{3}$ of 2 rt. ∠s $= \frac{2}{3}$ rt. ∠.

Hence, the ext. ∠ of an equiangular hexagon = the int. ∠ of an equiangular △.

Ex. **147**. If one diagonal of a quadrilateral bisects two of the angles. it is perpendicular to the other diagonal.



**Data:** Quadrilateral $ABCD$, such that the diagonal $BD$ bisects $\angle$s $ABC$ and $ADC$; also diagonal $AC$.

**To prove** $BD \perp AC$.

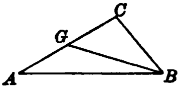
**Proof.** In $\triangle$s $ABD$ and $BCD$, $DB$ is common,

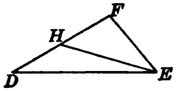
data, $\angle r = \angle s$, and $\angle t = \angle v$;

$\therefore$ § 102, $\triangle ABD = \triangle BCD$,

and, $AD = CD$; hence, § 90, $\triangle ACD$ is isosceles, and, § 120, $BD \perp AC$.

Ex. **148.** If one triangle has two sides and a median to one of them equal respectively to the corresponding parts of another triangle, the triangles are equal.





**Data:** Two $\triangle$s $ABC$ and $DEF$, in which $AC = DF$, $BC = EF$, and $BG$, the median to $AC$ equal to $EH$, the median to $DF$.

**To prove** $\triangle ABC = \triangle DEF$.

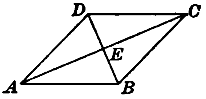
**Proof.** Data, $BC = EF$, $BG = EH$, and, data, Ax. 7, $GC = HF$;

$\therefore$ § 107, $\triangle GBC = \triangle HEF$, and $\angle C = \angle F$.

Then, data, $AC = DF$, $BC = EF$, and $\angle C = \angle F$;

$\therefore$ § 100, $\triangle ABC = \triangle DEF$.

Ex. **149.** The diagonals of a rhombus are unequal.



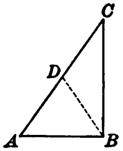
**Data:** Rhombus $ABCD$, and its diagonals $AC$ and $BD$.

**To prove** $AC$ and $BD$ unequal.

**Proof.** Of the $\angle$s $ADC$ and $DAB$, let $\angle ADC$ be the greater.

Then, in $\triangle$s $ADC$ and $DAB$, since the sides of a rhombus are equal, § 129, $AC > BD$.

Ex. **150.** If one angle of a triangle is equal to the sum of the other two, the triangle can be divided into two isosceles triangles.



**Data:** $\triangle ABC$, in which $\angle ABC = \angle A + \angle C$.

**To prove** $\triangle ABC$ can be divided into two isosceles $\triangle$s.

**Proof.** Draw $BD$ to meet $AC$ at $D$, making $\angle ABD = \angle A$.

Then, § 118, $\triangle ABD$ is isosceles.

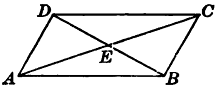
Data, $\angle A + \angle C = \angle ABC$,

Ax. 9, $\angle ABD + \angle DBC = \angle ABC$;

$\therefore$ $\angle A + \angle C = \angle ABD + \angle DBC$.

Const., $\angle A = \angle ABD$; $\therefore$ Ax. 3, $\angle C = \angle DBC$, and, § 118, $\triangle DBC$ is isosceles.

**Ex. 151.** If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.



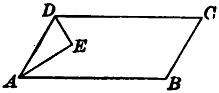
**Data:** Quadrilateral $ABCD$, whose diagonals, $AC$ and $BD$, bisect each other at $E$.

**To prove** $ABCD$ a parallelogram.

**Proof.** Data, $AE = CE$, $BE = DE$, and, § 59, $\angle AEB = \angle DEC$; $\therefore \triangle ABE = \triangle DCE$, and $\angle EAB = \angle ECD$: hence, § 75, $AB \parallel CD$.

Also, § 108, $AB = CD$; hence, § 150, $ABCD$ is a parallelogram.

**Ex. 152.** The bisectors of two adjacent angles of a parallelogram intersect each other at right angles.



**Data:** $\square ABCD$, and the bisectors, $AE$ and $DE$, of the adjacent $\angle s$ $BAD$ and $ADC$, produced to meet at $E$.

**To prove** $\angle AED$ a rt. $\angle$.

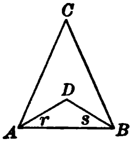
**Proof.** § 78, $\angle BAD + \angle ADC = 2$ rt. $\angle s$;

$\therefore$ Ax. 7, $\angle DAE + \angle ADE = 1$ rt. $\angle$;

but, § 110, $\angle DAE + \angle ADE + \angle AED = 2$ rt. $\angle s$;

$\therefore$ Ax. 3, $\angle AED = 1$ rt. $\angle$.

**Ex. 153.** If the bisectors of the equal angles of an isosceles triangle are produced until they meet, they form with the base an isosceles triangle.

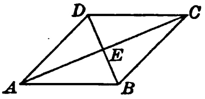


**Data:** Isos. $\triangle ABC$, in which $\angle CAB = \angle ABC$, and the bisectors, $AD$ and $BD$, of those $\angle s$, forming with the base $\triangle ABD$.

**To prove** $\triangle ABD$ isosceles.

**Proof.** Data, $\angle CAB = \angle ABC$; $\therefore$ data and Ax. 7, $\angle r = \angle s$; hence, § 118, $\triangle ABD$ is isosceles.

**Ex. 154.** The diagonals of a rhombus bisect the opposite angles.



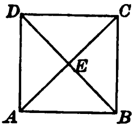
**Data:** Rhombus $ABCD$ and its diagonals $AC$ and $BD$.

**To prove** $AC$ and $BD$ bisect the opposite $\angle s$ of $ABCD$.

**Proof.** § 144, $AD = DC$; $\therefore$ § 90, $\triangle ACD$ is isosceles, and, Ex. 111, $BD$ bisects $AC$ at rt. $\angle s$; that is, $BD \perp AC$ at its middle point; hence, § 121, $BD$ bisects $\angle ADC$; similarly, $BD$ bisects $\angle ABC$. In like manner, $AC$ bisects $\angle s$ $BAD$ and $BCD$.

**Ex. 155.** If two equal straight lines bisect each other at right angles, the lines joining their extremities form a square.

**Data**: Equal lines, $AC$ and $BD$, bisecting each other at right angles at point $E$.



**To prove** $ABCD$ a square.

**Proof.** Data, Ax. 7, $AE = BE = CE = DE$,

and, § 52, $\angle AEB = \angle BEC = \angle CED = \angle AED$;

$\therefore$ § 100, $\triangle AEB = \triangle BEC = \triangle CED = \triangle AED$,

and $AB = BC = CD = DA$.

§ 111, $\angle ABE + \angle BAE = 1$ rt. $\angle$; but, § 108, $\angle BAE = \angle EBC$;

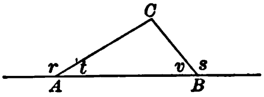
$\therefore$ substituting, $\angle ABE + \angle EBC = 1$ rt. $\angle$; that is, $\angle ABC$ is a rt. $\angle$.

Similarly, $\angle$s $BCD$, $ADC$, and $BAD$ are rt. $\angle$s.

Now, § 26, $AB \perp BC$, and $DC \perp BC$; $\therefore$ § 71, $AB \parallel DC$; similarly, $AD \parallel BC$.

Hence, § 143, $ABCD$ is a square.

**Ex. 156.** If the base of any triangle is produced in both directions, the sum of the two exterior angles diminished by the vertical angle is equal to two right angles.



**Data**: $\triangle ABC$ whose base is produced in both directions, forming the ext. $\angle$s $r$ and $s$.

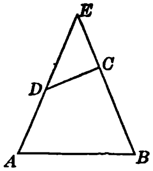
**To prove** $\angle r + \angle s - \angle C = 2$ rt. $\angle$s.

**Proof.** § 55, Ax. 2, $\angle r + \angle t + \angle v + \angle s = 4$ rt. $\angle$s;

but $\angle t + \angle v + \angle C = 2$ rt. $\angle$s;

hence, Ax. 3, $\angle r + \angle s - \angle C = 2$ rt. $\angle$s.

**Ex. 157.** In a quadrilateral, if two opposite sides which are not parallel are produced to meet, the perimeter of the greater triangle thus formed is greater than the perimeter of the quadrilateral.



**Data**: Quad. $ABCD$ whose sides $AD$ and $BC$ are produced to meet at $E$.

**To prove** perimeter of $\triangle ABE >$ perimeter $ABCD$.

**Proof.** § 125, $CE + DE > CD$;

adding $AB + BC + AD$ to each member,

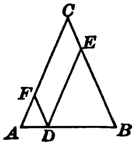
Ax. 4, $AB+BC+CE+AD+DE > AB+BC+CD+AD$,

or $AB + BE + AE > AB + BC + CD + AD$;

that is, perimeter of $\triangle ABE >$ perimeter $ABCD$.

**Ex. 158.** If from any point in the base of an isosceles triangle lines parallel to the sides are drawn, a parallelogram is formed whose perimeter is equal to the sum of the equal sides of the triangle.

**Data**: Isos. $\triangle ABC$; $D$ a point in the base; and lines $DF$ and $DE$ parallel to $BC$ and $AC$ respectively, meeting $AC$ and $BC$ in $F$ and $E$ respectively.



**To prove** $FDEC$ a $\square$ whose perimeter $= AC + BC$.

**Proof.** Data, $DF \parallel EC$, and $DE \parallel FC$; $\therefore$ § 140, $FDEC$ is a parallelogram.

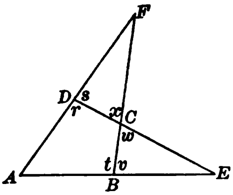
§ 76, $\angle EDB = \angle A$, § 116, $\angle B = \angle A$; $\therefore$ Ax. 1, $\angle EDB = \angle B$.

Hence, § 118, $DE = BE$; similarly, $DF = AF$;

$\therefore$ Ax. 2, $DF + DE + EC + FC = AF + FC + BE + EC$;

that is, perimeter of $FDEC = AC + BC$.

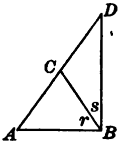
Ex. **159**. $ABCD$ is a quadrilateral having angle $ABC$ equal to angle $ADC$; $AB$ and $DC$ produced meet in $E$; $AD$ and $BC$ in $F$. Show that angle $AED$ equals angle $AFB$.



**Proof.** §§ 55, 32, $\angle v$ is sup. of $\angle t$, and $\angle s$ is sup. of $\angle r$; but, data, $\angle t = \angle r$; $\therefore$ § 54, $\angle v = \angle s$.

Then, in $\triangle$s $BEC$ and $DCF$, $\angle v = \angle s$, § 59, $\angle w = \angle x$; $\therefore$ § 113, $\angle E = \angle F$; that is, $\angle AED = \angle AFB$.

Ex. **160**. $ABC$ is an isosceles triangle having $AC$ equal to $BC$, and $AC$ is produced through $C$ its own length to $D$. Then, $ABD$ is a right angle.



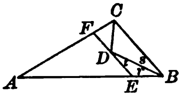
**Proof.** Data, $AC = BC = DC$;

$\therefore$ § 116, $\angle A = \angle r$, and $\angle D = \angle s$;

hence, Ax. 2, $\angle A + \angle D = \angle r + \angle s = \angle ABD$.

Therefore, Ex. 80, $\angle ABD$ is a rt. $\angle$.

Ex. **161**. $ABC$ is a triangle, and through $D$, the intersection of the bisectors of the angles $B$ and $C$, $EDF$ is drawn parallel to $BC$, meeting $AB$ in $E$ and $AC$ in $F$. Then, $EF = EB + FC$.



**Proof.** Data, $\angle r = \angle s$, § 73, $\angle t = \angle s$; $\therefore$ Ax. 1, $\angle r = \angle t$; hence, § 118, $ED = EB$.

Similarly, $DF = FC$; hence, Ax. 2, $ED + DF = EB + FC$, or $EF = EB + FC$.

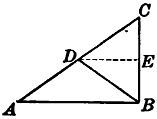
Ex. **162**. $ABCD$ is an isosceles trapezoid, and $AC$ and $BD$ its diagonals intersecting at $O$. Prove that the following pairs of triangles are equal: (1) $ABC$ and $ABD$; (2) $ADC$ and $BDC$; (3) $AOD$ and $BOC$.

**Proof.** (1) Data, $BC = AD$, $AB$ is common, and, Ex. 138, $AC = BD$; $\therefore$ § 107, $\triangle ABC = \triangle ABD$.

(2) Data, $AD = BC$, $DC$ is common, and, Ex. 138, $AC = BD$; $\therefore$ § 107, $\triangle ADC = \triangle BDC$.

(3) Ex. 142, ∆s $AOB$ and $DOC$ are isosceles; hence, in ∆s $AOD$ and $BOC$, § 90, $AO = BO$, $DO = CO$, and, data, $AD = BC$; ∴ § 107, $\triangle AOD = \triangle BOC$.

**Ex. 163.** In any right triangle the line drawn from the vertex of the right angle to the middle of the hypotenuse is equal to one half the hypotenuse.



**Data:** Rt. $\triangle ABC$, and $BD$ a line from the vertex of the rt. $\angle$ to the middle point of the hypotenuse $AC$.

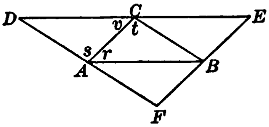
**To prove** $BD = \frac{1}{2} AC$.

**Proof.** Draw $DE \parallel AB$, and meeting $BC$ in $E$.

Data, $BC \perp AB$; ∴ § 72, $BC \perp DE$; that is, $DE \perp BC$, also, § 158, $BE = EC$; hence, § 132, $BD = DC$; but, data, $DC = \frac{1}{2} AC$; ∴ $BD = \frac{1}{2} AC$.

**Ex. 164.** If through each of the vertices of a triangle a line is drawn parallel to the opposite side, a new triangle is formed equal to four times the given triangle.

**Data:** $\triangle ABC$ and the lines $FD$, $FE$, and $DE$ drawn through $A$, $B$, and $C$ parallel to $BC$, $AC$, and $AB$ respectively, forming $\triangle DEF$.



**To prove** $\triangle DEF = 4 \triangle ABC$.

**Proof.** In ∆s $ACD$ and $ABC$, $AC$ is common,

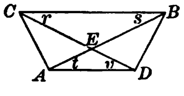
§ 73, $\angle s = \angle t$, and $\angle v = \angle r$;

∴ § 102, $\triangle ACD = \triangle ABC$.

Similarly, $\triangle AFB = \triangle ABC$, and $\triangle BCE = \triangle ABC$.

Hence, $\triangle DEF = \triangle ABC + \triangle ACD + \triangle AFB + \triangle BCE = 4 \triangle ABC$.

**Ex. 165.** Two equal lines, $AB$ and $CD$, intersect at $E$, and the triangles $CAE$ and $BDE$ are equal. Show that $CB$ is parallel to $AD$.



**Proof.** Since, data, $\triangle CAE = \triangle BDE$,

§ 108, $CE = BE$, and $AE = DE$,

and ∆s $CEB$ and $ADE$ are isosceles;

∴ § 116, $\angle r = \angle s$, and $\angle t = \angle v$.

Then, § 110, $2 \angle r + \angle CEB = 2$ rt. ∠s, and $2 \angle v + \angle AED = 2$ rt. ∠s;

∴ $2 \angle r + \angle CEB = 2 \angle v + \angle AED$; but, § 59, $\angle CEB = \angle AED$;

∴ Ax. 3, $2 \angle r = 2 \angle v$, or $\angle r = \angle v$; hence, § 75, $CB \parallel AD$.

**Ex. 166.** $ABC$ and $ABD$ are two equilateral triangles on opposite sides of the same base; $BE$ and $BF$ are the bisectors respectively of the angles $ABC$ and $ABD$, meeting $AC$ and $AD$ in $E$ and $F$ respectively. Then, the triangle $BEF$ is equilateral.

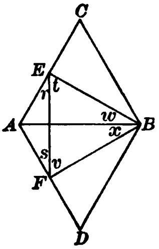
**Proof.** Ax. 1, $AC = AD$, § 120, $AE = \frac{1}{2} AC$, and $AF = \frac{1}{2} AD$;

∴ Ax. 7, $AE = AF$; hence, § 116, $\angle r = \angle s$;

but, § 120, $\angle$s $AEB$ and $AFB$ are rt. $\angle$s;

∴ § 31, $\angle t$ is comp. $\angle r$, and $\angle v$ is comp. $\angle s$;

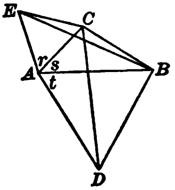
hence, § 54, $\angle t = \angle v$.



Ex. 74, $\angle ABC = 60°$; ∴ data, $\angle w = 30°$; similarly, $\angle x = 30°$; ∴ $\angle EBF = \angle w + \angle x = 30° + 30° = 60°$.

Since $\angle t = \angle v$, § 110, $2\angle t = 180° - \angle EBF = 120°$, and $\angle t = 60°$; consequently, $\angle v = 60°$; ∴ $\triangle BEF$ is equiangular; hence, § 119, $\triangle BEF$ is equilateral.

Ex. **167.** $ABC$ is any triangle, and on $AB$ and $AC$ equilateral triangles $ADB$ and $AEC$ are constructed externally. Show that $CD$ equals $BE$.



**Proof.** Ex. 74, $\angle t = 60°$, and $\angle r = 60°$;

∴ $\angle t = \angle r$; hence, Ax. 2, $\angle t + \angle s = \angle r + \angle s$,

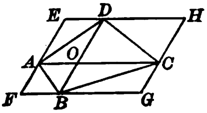
or, Ax. 9, $\angle CAD = \angle EAB$;

also, data, $AC = AE$, and, $AD = AB$;

∴ § 100, $\triangle ADC = \triangle ABE$, and $CD = BE$.

Ex. **168.** If through the extremities of each diagonal of a quadrilateral lines parallel to the other diagonal are drawn, a parallelogram double the given quadrilateral will be formed.

**Data:** Quad. $ABCD$, and its diagonals $AC$ and $BD$; $FE$ and $GH$ through $A$ and $C$ resp. and parallel to $BD$; and $FG$ and $EH$ through $B$ and $D$ resp. and parallel to $AC$, forming the figure $EFGH$.



**To prove** $EFGH$ a ▱ $= 2\,ABCD$.

**Proof.** Data, § 80, $FE \parallel GH$, and $FG \parallel EH$; ∴ § 140, $EFGH$ is a ▱.

Similarly, $AFBO$ is a ▱, and $AB$ is its diagonal;

∴ § 152, $\triangle AFB = \triangle ABO$,

and $AFBO = 2\triangle ABO$.

Similarly, $BGCO = 2\triangle BCO$,

$OCHD = 2\triangle OCD$,

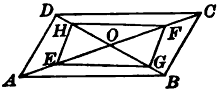
and $AODE = 2\triangle AOD$.

Adding, $AFBO + BGCO + OCHD + AODE$

$= 2(\triangle ABO + \triangle BCO + \triangle OCD + \triangle AOD)$;

that is, $EFGH = 2\,ABCD$.

**Ex. 169.** *ABCD* is a parallelogram; *E* and *F* are points on *AC*, such that $AE = FC$; *G* and *H* are points on *BD*, such that $BG = HD$. Then, *EGFH* is a parallelogram.



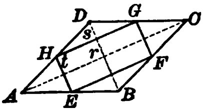
**Proof.** Let *AC* and *BD* intersect at *O*.

Then, § 154, $AO = OC$; but, data, $AE = FC$; $\therefore$ Ax. 3, $EO = FO$; similarly, $GO = HO$; and, § 59, $\angle EOG = \angle HOF$; $\therefore$ $\triangle EGO = \triangle HFO$, and $\angle OEG = \angle OFH$.

Hence, § 75, $EG \parallel HF$, and similarly, $EH \parallel GF$; $\therefore$ § 140, *EGFH* is a ▱.

**Ex. 170.** The lines joining the middle points of the sides of a rhombus taken in order form a rectangle.

**Data:** Rhombus *ABCD*, and the lines *EF*, *FG*, *GH*, and *HE* joining the middle points of its sides.



**To prove** *EFGH* a rectangle.

**Proof.** Since a rhombus is a parallelogram, Ex. 95, *EFGH* is a parallelogram.

Draw the diagonals *AC* and *DB*.

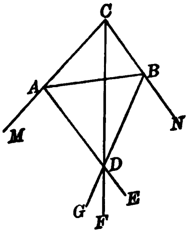
Then, in $\triangle ACD$, § 159, $HG \parallel AC$, similarly, $HE \parallel DB$; $\therefore$ § 73, $\angle t = \angle s$, and § 76, $\angle r = \angle s$; hence, Ax. 1, $\angle t = \angle r$; but, Ex. 111, $\angle r$ is a rt. $\angle$; $\therefore$ $\angle t$ is a rt. $\angle$.

Similarly, the other angles of *EFGH* are rt. $\angle$s.

Hence, § 141, *EFGH* is a rectangle.

**Ex. 171.** The bisector of the vertical angle of a triangle, and the bisectors of the exterior angles at the base formed by producing the sides about the vertical angle, meet in a point which is equidistant from the base and the sides produced.

**Data:** $\triangle ABC$, whose sides *CA* and *CB* are produced to *M* and *N* respectively, and the bisectors *AE*, *CF*, and *BG* of $\angle$s *MAB*, *ACB*, and *ABN* respectively.



**To prove** that *AE*, *CF*, and *BG* meet in a point which is equidistant from *AB*, *CM*, and *CN*.

**Proof.** Let any two of the bisectors, as *AE* and *CF*, intersect as in *D*.

Then, § 134, *D* is equidistant from *AB* and *CM*, and also from *CM* and *CN*.

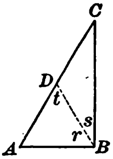
Hence, *D* is equidistant from *AB* and *CN*; $\therefore$ § 135, *D* lies in the bisector of $\angle ABN$.

That is, *BG* passes through the point *D* which is equidistant from *AB*, *CM*, and *CN*.

Ex. **172.** If in a right triangle one of the acute angles is twice the other, the hypotenuse is equal to twice the side opposite the smaller acute angle.

**Data:** Rt. $\triangle ABC$ in which $\angle A = 2\angle C$.

**To prove** $AC = 2\,AB$.



**Proof.** Draw $BD$ making $\angle r = \angle A$, and meeting $AC$ in $D$.

Data, $\angle A = 2\angle C$, and, § 111, $\angle A + \angle C = 3\angle C = 90°$; $\therefore \angle C = 30°$, and $\angle A = 2\angle C = 60°$.

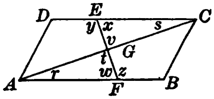
Const., $\angle r = \angle A = 60°$; $\therefore \angle s = 90° - \angle r = 30°$, and $\angle t = 180° - (\angle A + \angle r) = 180° - (60° + 60°) = 60°$.

Hence, § 118, $AD = BD = DC$, and $AD = \frac{1}{2}AC$; but, § 119, $AD = AB$; $\therefore AB = \frac{1}{2}AC$.

Ex. **173.** A parallelogram is bisected by any straight line passing through the middle point of one of its diagonals.

**Data:** $\square ABCD$ and the straight line $EF$ through $G$ the middle point of diagonal $AC$, meeting $DC$ in $E$ and $AB$ in $F$.

**To prove** $AFED = EFBC$.



**Proof.** Data, $AG = CG$, § 73, $\angle r = \angle s$, and, § 59, $\angle t = \angle v$; $\therefore$ § 102, $\triangle AFG = \triangle ECG$, and $AF = EC$.

§ 153, Ax. 3, $DE = FB$, § 153, $AD = BC$, and $EF$ is common to $AFED$ and $EFBC$.

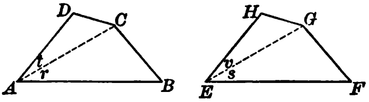
Also, § 73, $\angle w = \angle x$, and $\angle y = \angle z$; § 149, $\angle ADE = \angle FBC$, and $\angle FAD = \angle BCE$.

Hence, $AFED$ and $EFBC$ have their corresponding parts equal each to each, and may be made to coincide; therefore, § 36, $AFED = EFBC$.

Ex. **174.** If two quadrilaterals have three sides and the two included angles of one equal, each to each, to three sides and the two included angles of the other, the quadrilaterals are equal.

**Data:** Quadrilaterals $ABCD$ and $EFGH$, in which $AD = EH$, $CD = GH$, $BC = FG$, $\angle D = \angle H$, and $\angle BCD = \angle FGH$.

**To prove** $ABCD = EFGH$.



**Proof.** Draw the corresponding diagonals $AC$ and $EG$.

Data, § 100, $\triangle ACD = \triangle EGH$, and $\angle ACD = \angle EGH$; but, data, $\angle BCD = \angle FGH$; $\therefore$ Ax. 3, $\angle BCD - \angle ACD = \angle FGH - \angle EGH$, or $\angle ACB = \angle EGF$; also, § 108, $AC = EG$, and, data, $BC = FG$; $\therefore \triangle ABC = \triangle EFG$, $AB = EF$, $\angle B = \angle F$, and $\angle r = \angle s$; but, § 108, $\angle t = \angle v$; $\therefore$ Ax. 2, $\angle BAD = \angle FEH$.

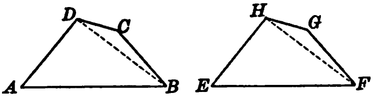
Hence, $ABCD$ and $EFGH$ have their corresponding parts equal, each to each, and they may be made to coincide.

Therefore, § 36, $ABCD = EFGH$.

**Ex. 175.** If two quadrilaterals have three angles and the two included sides of one equal, each to each, to three angles and the two included sides of the other, the quadrilaterals are equal.

**Data:** Quadrilaterals $ABCD$ and $EFGH$, in which $\angle ABC = \angle EFG$, $\angle C = \angle G$, $\angle ADC = \angle EHG$, $BC = FG$, and $CD = GH$.

**To prove** $ABCD = EFGH$.



**Proof.** Draw the corresponding diagonals $BD$ and $FH$.

Data, § 100, $\triangle BCD = \triangle FGH$, and $BD = FH$; data, $\angle ABC = \angle EFG$, and, § 108, $\angle DBC = \angle HFG$; $\therefore$ Ax. 3, $\angle ABC - \angle DBC = \angle EFG - \angle HFG$, or $\angle ABD = \angle EFH$; similarly, $\angle ADB = \angle EHF$; $\therefore$ § 102, $\triangle ABD = \triangle EFH$, $AB = EF$, $AD = EH$, and $\angle A = \angle E$.

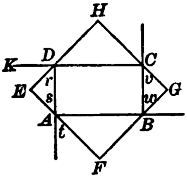
Hence, $ABCD$ and $EFGH$ have their corresponding parts equal, each to each, and they may be made to coincide.

Therefore, § 36, $ABCD = EFGH$.

**Ex. 176.** The bisectors of the exterior angles of a rectangle form a square.

**Data:** Rectangle $ABCD$, the bisectors of whose ext. $\angle$s meet to form the figure $EFGH$.

**To prove** $EFGH$ a square.



**Proof.** Since, § 141, $\angle ADC$ is a rt. $\angle$, $\angle ADK$ is a rt. $\angle$; $\therefore$ data, $\angle r = \frac{1}{2}$ rt. $\angle$; similarly, $\angle t = \frac{1}{2}$ rt. $\angle$; but, § 59, $\angle s = \angle t$; $\therefore \angle s = \frac{1}{2}$ rt. $\angle$.

Since $\angle r + \angle s = 1$ rt. $\angle$, $\angle E$ is a rt. $\angle$.

Similarly, $\angle$s $F$, $G$, and $H$ are rt. $\angle$s.

Also, similarly as above, $\angle$s $v$ and $w$ are each equal to $\frac{1}{2}$ rt. $\angle$.

Hence, § 116, $\triangle$s $ADE$ and $BGC$ are isos. $\triangle$s, in which $\angle r = \angle v$, $\angle s = \angle w$, and, § 153, $AD = BC$; $\therefore$ § 102, the $\triangle$s are equal, and $AE = DE = BG = CG$.

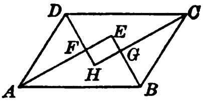
Similarly, $AF = FB = CH = DH$; hence, Ax. 2, $EF = FG = GH = HE$; § 26, $EH \perp EF$, and $FG \perp EF$; $\therefore$ § 71, $EH \parallel FG$, and similarly, $EF \parallel HG$.

Hence, § 143, $EFGH$ is a square.

**Ex. 177.** The bisectors of the interior angles of a parallelogram form a rectangle.

**Data:** $\square ABCD$, the bisectors of whose $\angle$s meet to form the figure $EFHG$.

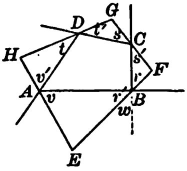
**To prove** $EFHG$ a rectangle.



**Proof.** Ex. 152, bisectors $CH$ and $DH$ intersect at rt. $\angle$s; $\therefore \angle FHG$ is a rt. $\angle$; similarly, the other angles of $EFHG$ are rt. $\angle$s.

Then, since $HG \perp FH$, and $FE \perp FH$, § 71, $HG \parallel FE$; similarly, $FH \parallel EG$; $\therefore$ § 140, $EFHG$ is a ▱, and since its $\angle$s are rt. $\angle$s, § 141, $EFHG$ is a rectangle.

Ex. **178**. The bisectors of the exterior angles of a quadrilateral form a quadrilateral whose opposite angles are supplementary.



**Data**: Quadrilateral $ABCD$, the bisectors of whose ext. $\angle$s meet to form quadrilateral $EFGH$.

**To prove** $\angle$s $E$ and $G$ supplementary and $\angle$s $F$ and $H$ supplementary.

**Proof.** § 167, sum of ext. $\angle$s of $ABCD = 4$ rt. $\angle$s;

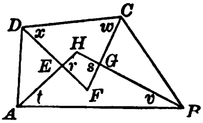
$\therefore$ data, Ax. 7, $\angle v + \angle r + \angle s + \angle t = 2$ rt. $\angle$s. (1)

Produce $CB$, then, Ex. 15, $\angle r' = \angle w$, and, § 59, $\angle r = \angle w$; $\therefore$ Ax. 1, $\angle r = \angle r'$; similarly, $\angle t = \angle t'$; hence, substituting in (1), $\angle v + \angle r' + \angle s + \angle t' = 2$ rt. $\angle$s; $\therefore$ $\angle v + \angle r'$ is supplementary to $\angle s + \angle t'$; but, § 110, $\angle E$ is supplementary to $\angle v + \angle r'$, and $\angle G$ is supplementary to $\angle s + \angle t'$.

Hence, $\angle$s $E$ and $G$ are supplementary.

Similarly, $\angle$s $F$ and $H$ are supplementary.

Ex. **179**. The bisectors of the interior angles of a quadrilateral form a quadrilateral whose opposite angles are supplementary.



**Data**: Quadrilateral $ABCD$, the bisectors of whose $\angle$s meet to form the quadrilateral $EFGH$.

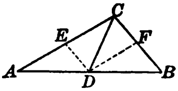
**To prove** $\angle$s $F$ and $H$ supplementary and $\angle$s $r$ and $s$ supplementary.

**Proof.** § 166, $\angle A + \angle B + \angle C + \angle D = 4$ rt. $\angle$s, data, Ax. 7, $\angle t + \angle v + \angle w + \angle x = 2$ rt. $\angle$s; $\therefore$ $\angle t + \angle v$ is supplementary to $\angle w + \angle x$; but, § 110, $\angle H + \angle t + \angle v = 2$ rt. $\angle$s, and $\angle H$ is supplementary to $\angle t + \angle v$; similarly, $\angle F$ is supplementary to $\angle w + \angle x$.

Hence, $\angle$s $F$ and $H$ are supplementary.

Similarly, $\angle$s $r$ and $s$ are supplementary.

Ex. **180**. The straight line drawn from any vertex of a triangle to the middle point of the opposite side is less than half the sum of the other two sides.



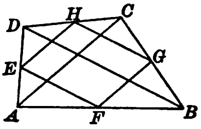
**Data**: $\triangle ABC$ and the median $CD$.

**To prove** $CD < \frac{1}{2}(AC + BC)$.

**Proof.** Draw $DF \parallel AC$, and $DE \parallel BC$, meeting $BC$ and $AC$ in $F$ and $E$ respectively.

Then, since $D$ is the middle point of $AB$, § 158, $DF = \frac{1}{2} AC$, and $CF = \frac{1}{2} BC$; but, § 124, $CD < DF + CF$; $\therefore$ substituting, $CD < \frac{1}{2} AC + \frac{1}{2} BC$; that is, $CD < \frac{1}{2}(AC + BC)$.

**Ex. 181.** The lines which join the middle points of the sides of a quadrilateral successively form a parallelogram whose perimeter is equal to the sum of the diagonals of the quadrilateral.



**Data:** Quadrilateral $ABCD$, whose diagonals are $AC$ and $BD$, and the lines $EF$, $FG$, $GH$, and $EH$ joining the middle points of the sides of $ABCD$.

**To prove** $EFGH$ a parallelogram whose perimeter $= AC + BD$.

**Proof.** In $\triangle ACD$, data, § 159, $EH \parallel AC$, similarly, $FG \parallel AC$; $\therefore$ § 80, $EH \parallel FG$.

Similarly, $EF \parallel HG$; $\therefore$ § 140, $EFGH$ is a $\square$.

Now, § 158, $EH = \frac{1}{2} AC$, $FG = \frac{1}{2} AC$, $EF = \frac{1}{2} BD$, and $HG = \frac{1}{2} BD$;

$\therefore$ Ax. 2, $EH + FG + EF + HG = \frac{1}{2} AC + \frac{1}{2} AC + \frac{1}{2} BD + \frac{1}{2} BD$;

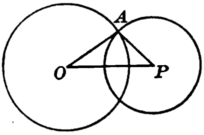
that is, perimeter of $EFGH = AC + BD$.

## BOOK II

Prop. III, page 88. (Converse.)

**Proof.** Ax. 14, $OA = PD$, $OB = PE$, and, § 194, $\angle O = \angle P$; $\therefore$ § 100, $\triangle OAB = \triangle PDE$, and chord $AB$ = chord $DE$.

**Ex. 182.** If two circumferences intersect, the distance between their centers is greater than the difference of their radii.



**Data:** Two circumferences whose centers are $O$ and $P$ intersecting at $A$, and the radii $OA$ and $PA$, $OA$ being the longer.

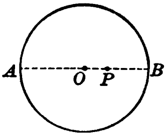
**To prove** $OP > OA - PA$.

**Proof.** Ex. 132, $OA - PA < OP$; that is, $OP > OA - PA$.

**Ex. 183.*** A straight line can cut a circumference in only two points.

**Proof.** Ax. 14, the straight lines drawn from the center of the ⊙ to the points of intersection are equal, and if there could be three points of intersection, there would be three equal straight lines drawn from a point to a straight line, which, § 133, is impossible; hence, a straight line can intersect a circumference in *only two* points.

**Ex. 184.** A circle can have but one center.



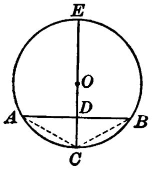
**Proof.** Suppose the ⊙ can have another center, as $P$. Draw the diameter $AB$ through $O$ and $P$.

Ax. 14, $OB = OA$, hyp. and Ax. 14, $PB = OA$; $\therefore$ Ax. 1, $OB = PB$; but this is impossible, for, Ax. 8, $OB > PB$.

Hence, $P$ is not a center and the supposition that the ⊙ can have a center other than $O$ is untenable.

* See page 312.

Ex. **185**. If a straight line bisects a chord and its subtended arc, it is perpendicular to the chord.



**Data:** Chord $AB$ and the line $EC$ bisecting $AB$ at $D$ and arc $AB$ at $C$.

**To prove** $EC \perp AB$.

**Proof.** Draw the chords $AC$ and $BC$.

Data, arc $AC =$ arc $BC$; $\therefore$ § 196, chord $AC =$ chord $BC$; also, data, $AD = BD$; that is, $C$ and $D$ are each equidistant from $A$ and $B$; hence, § 106, $CD \perp AB$; that is, $EC \perp AB$.

Ex. **186**. The perpendicular bisectors of the sides of an inscribed quadrilateral meet in a common point.

**Proof.** The sides of an inscribed figure are chords of the circle; hence, § 199, their perpendicular bisectors pass through the center of the circle.

That is, the perpendicular bisectors of the sides of an inscribed quadrilateral meet in a common point which is the center of the circle.

Ex. **187**. If a diameter of a circle bisects a chord, it is perpendicular to the chord and bisects its subtended arc.

**Data:** (See Fig. for Ex. 185.) Diameter $CE$ bisecting chord $AB$ at $D$.

**To prove** $CE \perp AB$, and bisects arc $AB$.

**Proof.** § 173, $O$ is equidistant from $A$ and $B$,

data, $D$ is equidistant from $A$ and $B$;

$\therefore$ § 106, $OD \perp AB$;

but, § 176 and data, $O$ and $D$ are each in $CE$;

$\therefore$ $CE \perp AB$.

Hence, § 200, $CE$ bisects arc $AB$.

Ex. **188**. If a diameter of a circle bisects an arc, it is perpendicular to the chord of the arc at its middle point.

**Data:** Chord $AB$ and diameter $CE$ bisecting arc $AB$ at $C$.

**To prove** $CE \perp AB$ at its middle point.

**Proof.** (See Fig. for Ex. 185.) Draw chords $AC$ and $BC$.

Data, arc $AC =$ arc $BC$; $\therefore$ § 196, chord $AC =$ chord $BC$;

that is, $C$ is equidistant from $A$ and $B$;

also, § 173, $O$ is equidistant from $A$ and $B$;

$\therefore$ § 106, $OC \perp AB$ at its middle pt.;

but, § 176 and data, $O$ and $C$ are each in $CE$;

hence, $CE \perp AB$ at its middle pt.

Ex. **189.** If the distance from the center of a circle to a straight line is less than the radius, the line will cut the circumference, and if the distance is greater than the radius, the line will not cut the circumference.

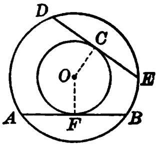
**Proof.** If the distance is less than the radius, the foot of the perpendicular from the center to the line will be within the circle, and the line will cut the circumference in two points ; but, if the distance is greater than the radius, the foot of the perpendicular from the center to the line will be without the circle, and since, § 61, it is nearer the center of the circle than any other point of the line, the line will not cut the circumference.

Ex. **190.** The line drawn through the middle points of two parallel chords in a circle passes through the center.

**Proof.** Erect a perpendicular to one of the chords at its middle point; then, § 199, this perpendicular passes through the center of the circle, and, §§ 72, 200, it is perpendicular to the other chord at its middle point, and, Ax. 11, must coincide with the line joining the middle points of the chords.

Hence, the line joining the middle points of two chords in a circle passes through the center.

Ex. **191.** If two circles are concentric, any two chords of the greater, which are tangent to the less, are equal.



**Data:** Two concentric ⓢ whose center is $O$, and two chords, as $AB$ and $DE$, of the greater ⊙, that are tangent to the less at $F$ and $C$ respectively.

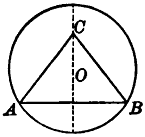
**To prove** $AB = DE$.

**Proof.** Since, Ax. 17, $AB$ has only one pt. $F$ in common with the smaller ⊙, its least distance from $O$ is the radius $OF$ of the smaller ⊙, and Ax. 14, $DE$ is at the same distance $OC$ from $O$.

Then, in the larger ⊙, § 202, $AB = DE$.

Ex. **192.** If an isosceles triangle is constructed on any chord of a circle as its base, the vertex lies on the diameter that is perpendicular to the chord, or on that diameter produced.

**Data :** Any isos. $\Delta$, as $ABC$, on the chord $AB$ as a base.



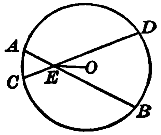
**To prove** vertex $C$ lies on the diameter that is perpendicular to $AB$, or on that diameter produced.

**Proof.** § 90, $C$ is equidistant from $A$ and $B$; $\therefore$ § 104, $C$ lies on the line perpendicular to $AB$ at its middle point; but, § 199, the line perpendicular to $AB$ at its middle point passes through the center of the circle, and is a diameter.

Hence, $C$ lies on the diam. $\perp AB$ or on that diam. produced.

Ex. **193.** If two chords of a circle cut each other and make equal angles with the straight line which joins their point of intersection with the center, the chords are equal.

**Data**: Two chords, $AB$ and $CD$, in the circle whose center is $O$, intersecting at $E$, and making equal angles with $OE$.



**To prove** $AB = CD$.

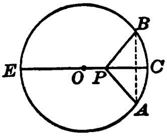
**Proof.** Data, $OE$ is the bisector of $\angle DEB$;

∴ § 134, $O$ is equidistant from $AB$ and $CD$;

that is, $AB$ and $CD$ are equally distant from $O$;

hence, § 202, $AB = CD$.

**Ex. 194.** If from any point within a circle two equal straight lines are drawn to the circumference, the bisector of the angle thus formed passes through the center of the circle.



**Data**: Point $P$ within a circle; two equal lines, $PA$ and $PB$, from $P$ to the circumference; and the line $CE$ bisecting $\angle APB$.

**To prove** that $CE$ passes through the center of the circle.

**Proof.** Draw chord $AB$.

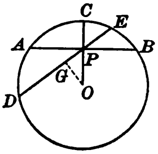
Then, data and § 90, $ABP$ is an isosceles △;

∴ § 120, $CE \perp AB$ at its middle point;

hence, § 199, $CE$ passes through $O$ the center of the circle.

**Ex. 195.** If in a circle a chord is perpendicular to a radius at any point, it is the shortest chord that can be drawn through that point.

**Data**: The circle whose center is $O$; the radius $OC$; chord $AB \perp OC$ at $P$; and any other chord through $P$, as $DE$.



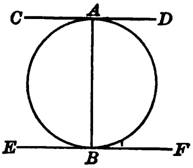
**To prove** $AB < DE$.

**Proof.** Draw $OG \perp DE$, meeting $DE$ in $G$.

Then, § 61, $OG < OP$; that is, $OP > OG$; but, § 62, $OG$ represents the distance from $O$ to $DE$, and $OP$ the distance from $O$ to $AB$.

Hence, § 204, $AB < DE$.

**Ex. 196.** If tangents are drawn through the extremities of a diameter, they are parallel to each other.



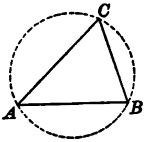
**Data**: $AB$ the diameter of a circle, and the tangents $CD$ and $EF$ at $A$ and $B$ respectively.

**To prove** $CD \parallel EF$.

**Proof.** § 205, $CD \perp AB$, and $EF \perp AB$;

∴ § 71, $CD \parallel EF$.

**Ex. 197.** Prove by Prop. XVII that the sum of the angles of a triangle is equal to two right angles.



**Data:** $\angle$s $A$, $B$, and $C$ of the $\triangle ABC$.

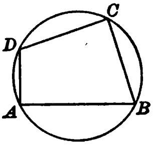
**To prove** $\angle A + \angle B + \angle C = 2$ rt. $\angle$s.

**Proof.** By § 207, pass a circumference through the vertices $A$, $B$, and $C$.

Then, § 225, $\angle A$ is measured by $\frac{1}{2}$ arc $BC$,
$\angle B$ is measured by $\frac{1}{2}$ arc $AC$,
and $\angle C$ is measured by $\frac{1}{2}$ arc $AB$;
hence, the sum of the measures of $\angle$s $A$, $B$, and $C = \frac{1}{2}(BC + AC + AB) = 180°$.

Consequently, $\angle A + \angle B + \angle C = 2$ rt. $\angle$s.

Ex. **198**. The opposite angles of an inscribed quadrilateral are supplementary.



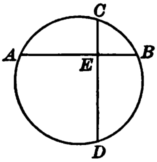
**Datum:** Inscribed quadrilateral $ABCD$.

**To prove** $\angle$s $A$ and $C$ supplementary and angles $B$ and $D$ supplementary.

**Proof.** § 225, $\angle A$ is measured by $\frac{1}{2}$ arc $BCD$,
and $\angle C$ is measured by $\frac{1}{2}$ arc $BAD$;
hence, the sum of the measures of $\angle$s $A$ and $C$ = a semicircumference = 180°.

Consequently, $\angle A + \angle C = 2$ rt. $\angle$s.
Hence, § 32, $\angle$s $A$ and $C$ are supplementary.
Similarly, $\angle$s $B$ and $D$ are supplementary.

Ex. **199**. If two chords of a circle intersect at right angles, the sum of any pair of opposite arcs equals a semicircumference.



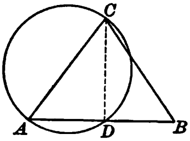
**Data:** Two chords, $AB$ and $CD$, intersecting at right angles in $E$.

**To prove** arc $AD$ + arc $BC$ = a semicircumference.

**Proof.**

§ 230, $\angle AED$ is measured by $\frac{1}{2}$(arc $AD$ + arc $BC$);
but, data, $\angle AED$ = a rt. $\angle = 90°$;
$\therefore$ $\frac{1}{2}$(arc $AD$ + arc $BC$) = 90°;
whence, arc $AD$ + arc $BC$ = 180° = a semicircumference.

Ex. **200**. If one of the equal sides of an isosceles triangle is the diameter of a circle, the circumference bisects the base.



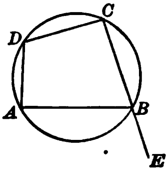
**Data:** Isos. $\triangle ABC$, one of whose equal sides, as $AC$, is the diameter of a circle whose circumference cuts the base $AB$ at $D$.

**To prove** $AD = BD$.

**Proof.** Draw $CD$.

Then, §§ 188, 192, $\angle ADC$ is inscribed in a semicircle;
$\therefore$ § 227, $\angle ADC$ is a rt. $\angle$;
hence, § 122, $AD = BD$.

**Ex. 201.** If one side of an angle of a quadrilateral inscribed in a circle is produced, the exterior angle is equal to the opposite angle of the quadrilateral.

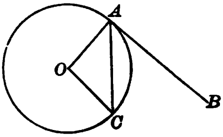


**Data**: Inscribed quadrilateral *ABCD*, whose side *CB* is produced to *E*.

**To prove** $\angle ABE = \angle D$.

**Proof.** §§ 55, 32, $\angle ABE$ is sup. of $\angle ABC$,
and, Ex. 198, $\angle D$ is sup. of $\angle ABC$;
hence, § 54, $\angle ABE = \angle D$.

**Ex. 202.** The angle between a tangent to a circle and a chord drawn from the point of contact is half the angle at the center subtended by that chord.

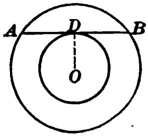


**Data**: The circle whose center is $O$; $AB$ tangent at $A$; chord $AC$; and the central $\angle AOC$.

**To prove** $\angle BAC = \frac{1}{2} \angle AOC$.

**Proof.** § 231, $\angle BAC$ is measured by $\frac{1}{2}$ arc $AC$,
and, § 224, $\angle AOC$ is measured by arc $AC$;
$\therefore$ the measure of $\angle BAC$ is $\frac{1}{2}$ the measure of $\angle AOC$;
consequently, $\angle BAC = \frac{1}{2} \angle AOC$.

**Ex. 203.** A line which is tangent to the inner of two concentric circles, and is a chord of the outer circle, is bisected at the point of tangency.

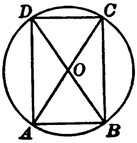


**Data**: Two concentric circles whose center is $O$, and the chord $AB$ of the outer, tangent to the inner at $D$.

**To prove** $AD = BD$.

**Proof.** Draw $OD$, the radius to the point of contact.
Then, § 205, $OD \perp AB$;
hence, § 200, $AD = BD$.

**Ex. 204.** The diagonals of a rectangle inscribed in a circle are diameters of the circle.

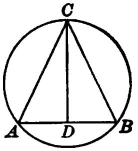


**Data**: Rectangle *ABCD* inscribed in a circle whose center is $O$, and the diagonals $AC$ and $BD$.

**To prove** $AC$ and $BD$ diameters of the circle.

**Proof.** § 225, $\angle ADC$ is measured by $\frac{1}{2}$ arc $ABC$;
but, data, $\angle ADC = 90°$;
consequently, arc $ABC = 180°$, or a semicircumference;
and, § 192, $AC$ is a diameter.
Similarly, $BD$ is a diameter.

**Ex. 205.** The bisector of the vertical angle of an inscribed isosceles triangle passes through the center of the circle.

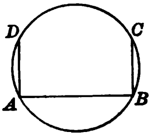


**Data:** Inscribed isos. $\triangle ABC$, and the bisector $CD$, of its vert. $\angle$, meeting $AB$ at $D$.

**To prove** $CD$ passes through the center of the circle.

**Proof.** § 120, $CD \perp AB$ at its middle point; hence, § 199, $CD$ passes through the center of the circle.

**Ex. 206.** Two chords perpendicular to a third chord at its extremities are equal.



**Data:** Two chords $AD$ and $BC$, perpendicular to chord $AB$ at its extremities $A$ and $B$ respectively.

**To prove** chord $AD$ = chord $BC$.

**Proof.** Since, § 52, $\angle$s $DAB$ and $ABC$ are equal,

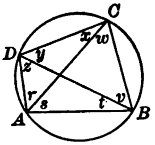
$$\text{arc } AD + \text{arc } DC = \text{arc } BC + \text{arc } DC.$$

Taking arc $DC$ from each member of this equation,

Ax. 3, $\quad$ arc $AD$ = arc $BC$;

hence, § 196, $\quad$ chord $AD$ = chord $BC$.

**Ex. 207.** If a quadrilateral is inscribed in a circle and its diagonals are drawn, the diagonals will divide the angles of the quadrilateral so that there will be four pairs of equal angles.



**Data:** Inscribed quad. $ABCD$, and its diagonals $AC$ and $BD$.

**To prove** $\angle r = \angle v$, $\angle s = \angle y$, $\angle t = \angle x$, and $\angle w = \angle z$.

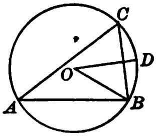
**Proof.** § 225, $\angle r$ is measured by $\frac{1}{2}$ arc $DC$

and $\quad$ $\angle v$ is measured by $\frac{1}{2}$ arc $DC$;

hence, $\quad$ $\angle r = \angle v$.

Similarly, $\quad$ $\angle s = \angle y$, $\angle t = \angle x$, and $\angle w = \angle z$.

**Ex. 208.** If from the center of a circle a perpendicular is drawn to either side of an inscribed triangle, and a radius is drawn to one end of this side, the angle between the radius and the perpendicular is equal to the opposite angle of the triangle.



**Data:** $\triangle ABC$, whose base is $AB$, inscribed in the circle whose center is $O$; $OD \perp BC$, meeting the circumference in $D$; and the radius $OB$.

**To prove** $\quad$ $\angle BOD = \angle BAC$.

**Proof.** § 224, $\angle BOD$ is measured by arc $BD$,

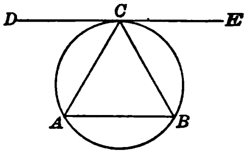
and, § 225, $\quad$ $\angle BAC$ is measured by $\frac{1}{2}$ arc $CDB$;

but, § 198, $\quad$ $OD$ bisects arc $CDB$; $\therefore$ $\frac{1}{2}$ arc $CDB$ = arc $BD$;

consequently, $\quad$ $\angle$s $BOD$ and $BAC$ have the same measure;

hence, $\quad$ $\angle BOD = \angle BAC$.

**Ex. 209.** The tangent at the vertex of an inscribed equilateral triangle forms equal angles with the adjacent sides.



**Data**: Inscribed equilateral $\triangle ABC$, and the tangent $DE$ at $C$.

**To prove** $\angle ACD = \angle BCE$.

**Proof.** § 91, chord $AC$ = chord $BC$;

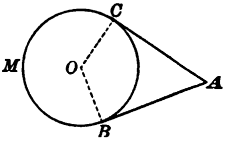
$\therefore$ § 196, arc $AC$ = arc $BC$.

Now, § 231, $\angle ACD$ is measured by $\frac{1}{2}$ arc $AC$,

and $\angle BCE$ is measured by $\frac{1}{2}$ arc $BC$;

hence, $\angle ACD = \angle BCE$.

**Ex. 210.** The angle between two tangents from the same point is 24° 15′. How many degrees are there in each of the intercepted arcs?



**Solution.** Let $AB$ and $AC$ be tangents to the circle whose center is $O$ and the $\angle A = 24° 15'$. Draw $OB$ and $OC$ to the points of contact.

§ 166, the sum of the $\angle$s of $ABOC$ = 4 rt. $\angle$s = 360°,

and, § 205, $\angle ABO + \angle ACO$ = 2 rt. $\angle$s = 180°;

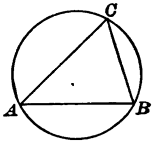
$\therefore$ Ax. 3, $\angle O + \angle A = 180°$;

hence, $\angle O = 180° - 24° 15' = 155° 45'$;

consequently, § 224, arc $BC = 155° 45'$,

and arc $BMC = 360° -$ arc $BC = 204° 15'$.

**Ex. 211.** One angle of an inscribed triangle is 38°, and one of its sides subtends an arc of 124°. What are the other angles of the triangle?



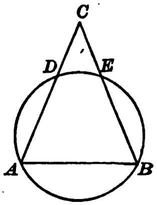
**Solution.** Let $ABC$ be an inscribed $\triangle$ whose $\angle A = 38°$ and whose side $AB$ subtends an arc of 124°.

§ 225, $\angle C$ is measured by $\frac{1}{2}$ arc $AB$;

hence, $\angle C = \frac{1}{2}$ of $124° = 62°$.

$\angle B = 180° - (\angle A + \angle C) = 180° - (38° + 62°) = 80°$.

**Ex. 212.** $AB$, a chord of a circle, is the base of an isosceles triangle whose vertex $C$ is without the circle, and whose equal sides intersect the circumference at $D$ and $E$. Prove that $CD$ is equal to $CE$.



**Proof.** Since, § 116, $\angle$s $B$ and $A$ are equal,

arc $AD$ + arc $DE$ = arc $BE$ + arc $DE$.

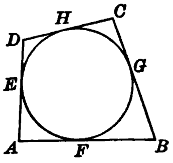
Taking arc $DE$ from each member of this equation,

Ax. 3, arc $AD$ = arc $BE$;

hence, § 196, chord $AD$ = chord $BE$;

$\therefore$ Ax. 3, $AC - AD = BC - BE$, or $CD = CE$.

Ex. **213.** In any quadrilateral circumscribing a circle any pair of opposite sides is equal to half the perimeter of the quadrilateral.



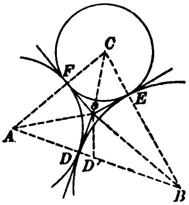
**Data:** Circumscribed quad. $ABCD$, whose sides are tangent to the circle at $E$, $F$, $G$, and $H$.

**To prove** $AB + DC = AD + BC$.

**Proof.** § 209, $AF = AE$, $FB = BG$, $HC = GC$, and $HD = DE$;

$\therefore$ Ax. 2, $AF + FB + HC + HD = AE + DE + BG + GC$;
that is, $AB + DC = AD + BC$.

Ex. **214.** If three circles touch each other externally, and the three common tangents are drawn, these tangents meet in a point equidistant from the points of contact of the circles.



**Data:** Three Ⓢ whose centers **are** $A$, $B$, **and** $C$, tangent externally at $D$, $E$, and $F$.

**To prove** that the common tangents through $D$, $E$, and $F$ meet in a pt. equidistant from $D$, $E$, and $F$.

**Proof.** Draw $\triangle ABC$ and find $O$, the intersection of its angle bisectors. Draw $OD$, $OE$, and $OF$.

§ 100, $\triangle OBD = \triangle OBE$; $\triangle OEC = \triangle OFC$; and $\triangle OFA = \triangle ODA$. Whence, $OD = OE = OF$; also $\angle ODB = \angle OEB$, $\angle OEC = \angle OFC$, and $\angle OFA = \angle ODA$.

If $OD$ is not $\perp$ to $AB$, draw $OD' \perp AB$ and let $\angle DOD' = x$.

If $D'$ lies between $D$ and $B$, $\angle ODB = 1$ rt. $\angle - x$, $\angle OEB = 1$ rt. $\angle - x$, $\angle OEC = 1$ rt. $\angle + x$, $\angle OFC = 1$ rt. $\angle + x$, $\angle OFA = 1$ rt. $\angle - x$, and $\angle ODA = 1$ rt. $\angle - x$.

Adding, § 55, 6 rt. $\angle$s $= 6$ rt. $\angle$s $- 2x$, which is absurd, and $D'$ does not lie between $D$ and $B$. Similarly, $D'$ does not lie between $A$ and $D$.

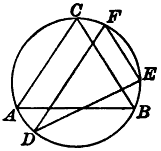
Hence, $D'$ must coincide with $D$; $OD \perp AB$ at $D$; and § 205, $OD$ is the common tangent through $D$.

Similarly, $OE$ and $OF$ are the common tangents through $E$ and $F$.

Ex. **215.** If two triangles are inscribed in a circle, and two sides of one are parallel, each to each, to two sides of the other, the third sides are equal.

**Data:** Inscribed $\triangle$s $ABC$ and $DEF$ such that $AC \parallel DF$ and $BC \parallel EF$.

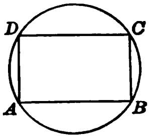
**To prove** $AB = DE$.



**Proof.** § 225, $\angle C$ is meas. by $\frac{1}{2}$ arc $AB$, $\angle F$ by $\frac{1}{2}$ arc $DE$; but since, data, the sides of $\angle C$ are parallel to the sides of $\angle F$, § 81, $\angle C = \angle F$; or $\angle$s $C$ and $F$ are supplementary; when $\angle C = \angle F$, the measures of $\angle$s $C$ and $F$ are equal, and, Ax. 6, arc $AB$ = arc $DE$; hence, § 196, $AB = DE$.

When $\angle$s inscr. in the same ⊙ are sup., the arcs which subtend them together = a circum. $\therefore$ the same chord may subtend both arcs; *i.e.* the chords which subtend the arcs intercepted by the sides of sup. $\angle$s are equal.

**Ex. 216.** Every parallelogram inscribed in a circle is a rectangle.



**Datum:** Any inscribed ▱, as $ABCD$.

**To prove** $ABCD$ a rectangle.

**Proof.** § 140, $AB \parallel DC$, and $AD \parallel BC$;

$\therefore$ § 206, arc $AD =$ arc $BC$, and arc $AB =$ arc $DC$;

hence, Ax. 2, arc $AD +$ arc $AB =$ arc $BC +$ arc $DC$,

or arc $BAD =$ arc $BCD$;

but arc $BAD +$ arc $BCD = 360°$; $\therefore$ arc $BAD = 180°$.

Hence, § 225, $\angle C$ is measured by $\frac{1}{2}$ arc $BAD$, or $90°$;

consequently, $\angle C$ is a rt. $\angle$.

Similarly, $\angle$s $A$, $D$, and $B$ are rt. $\angle$s, and, § 141, $ABCD$ is a rectangle.

**Ex. 217.** Two sides of an inscribed triangle subtend $\frac{1}{5}$ and $\frac{1}{6}$ of the circumference, respectively. What are the angles of the triangle?

**Solution.** Let $ABC$ be an inscribed $\triangle$ whose side $AC$ subtends $\frac{1}{6}$ of the circum., and whose side $AB$ subtends $\frac{1}{5}$ of the circum.

Then, § 225, $\angle B$ is measured by $\frac{1}{2}$ arc $AC$, or $\frac{1}{2}$ of $\frac{1}{6}$ of $360°$;

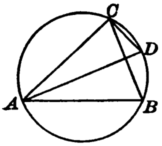
hence, $\angle B = 30°$.

Also, $\angle C$ is measured by $\frac{1}{2}$ arc $AB$, or $\frac{1}{2}$ of $\frac{1}{5}$ of $360°$;

hence, $\angle C = 36°$.

§ 110, $\angle A = 180° - (\angle B + \angle C) = 180° - 66° = 114°$.

**Ex. 218.** If a circle is circumscribed about the triangle $ABC$, and a line is drawn bisecting angle $A$ and meeting the circumference in $D$, angle $DCB$ is equal to one half angle $BAC$.



**Proof.** § 225, $\angle BCD$ is measured by $\frac{1}{2}$ arc $BD$,

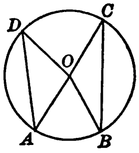
and $\angle BAD$ is measured by $\frac{1}{2}$ arc $BD$;

hence, $\angle BCD = \angle BAD$;

but, data, $\angle BAD = \frac{1}{2} \angle BAC$;

hence, Ax. 1, $\angle BCD = \frac{1}{2} \angle BAC$.

**Ex. 219.** $AB$ is an arc of 65°, $DC$ an arc of 75° in a circle whose center is $O$. $AC$ is a diameter. How many degrees are there in each angle of the triangles $AOD$ and $BOC$?



**Solution.** § 192, arc $AD +$ arc $DC = 180°$; hence, arc $AD = 180° - 75° = 105°$; $\therefore$ § 224, $\angle AOD = 105°$.

§ 225, $\angle DAC$ is measured by $\frac{1}{2}$ arc $DC$, or $37\frac{1}{2}°$; $\therefore \angle DAC = 37\frac{1}{2}°$, and since, Ax. 14, $OD = OA$, § 116, $\angle ADO = \angle DAO = 37\frac{1}{2}°$.

Also, § 192, arc $AB +$ arc $BC = 180°$;

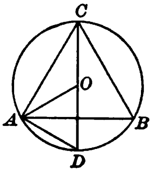
hence, arc $BC = 180° - 65° = 115°$;

$\therefore$ § 224, $\angle BOC = 115°$.

§ 225, $\angle ACB$ is measured by $\frac{1}{2}$ arc $AB$, or $32\frac{1}{2}°$; $\therefore \angle ACB = 32\frac{1}{2}°$, and since, Ax. 14, $OB = OC$, § 116, $\angle OBC = \angle OCB = 32\frac{1}{2}°$.

**Ex. 220.** If an equilateral triangle is inscribed in a circle, and a diameter is drawn from one vertex, the triangle formed by joining the other extremity of the diameter and the center of the circle with one of the other vertices of the inscribed triangle is also equilateral.

**Data:** Inscribed equilateral $\triangle ABC$; diameter $CD$; $O$ the center of the circle, and the lines $OA$ and $AD$.



**To prove** $\triangle ADO$ equilateral.

**Proof.** Data and § 173, $C$ and $O$ are each equidistant from $A$ and $B$; hence, § 106, $CD \perp AB$; $\therefore$ § 200, arc $AD = \frac{1}{2}$ arc $AB$.

Now, Ex. 74, $\angle ACB = 60°$;

but $\angle ACB$ is measured by $\frac{1}{2}$ arc $AB$;

$\therefore$ arc $AD = \frac{1}{2}$ arc $AB = 60°$;

consequently, § 224, $\angle AOD = 60°$.

Since, Ax. 14, $AO = OD$, § 116, $\angle OAD = \angle ADO$;

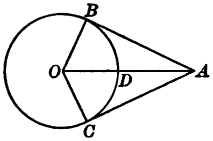
$\therefore$ § 110, $2\angle ADO = 180° - 60° = 120°$, and $\angle ADO = 60°$, also $\angle OAD = 60°$

Consequently, $\triangle ADO$ is equiangular.

Hence, § 119, $\triangle ADO$ is equilateral.

**Ex. 221.** If tangents are drawn to a circle from a point without, the line joining that point with the center of the circle bisects (1) the angle formed by the tangents; (2) the angle formed by the radii drawn to the points of tangency; and (3) the arc intercepted by these radii.

**Data:** The ⊙ whose center is $O$; pt. $A$ without it; $AB$ and $AC$ tangents at $B$ and $C$ respectively; radii $OB$ and $OC$; and line $AO$ cutting the circum. at $D$.



**To prove**
(1) $\angle BAO = \angle CAO$,
(2) $\angle AOB = \angle AOC$,
(3) arc $BD$ = arc $CD$.

**Proof.** (1) In $\triangle$s $ABO$ and $ACO$, $AO$ is common,

§ 209, $AB = AC$, and, Ax. 14, $OB = OC$;

$\therefore$ § 107, $\triangle ABO = \triangle ACO$, and $\angle BAO = \angle CAO$.

(2) § 108, $\angle AOB = \angle AOC$.

(3) § 194, arc $BD$ = arc $CD$.

PROP. XXVI, page 113.

**Proof. Case I.** Const., $F$ and $C$ are each equidistant from $D$ and $E$; hence, § 106, $FC \perp DE$; that is, $FC \perp AB$.

**Case II.** Const., and § 176, $HK$ is a diam.; $\therefore$ § 192, $HBK$ is a semicircle; hence, § 227, $\angle HBK$ is a rt. $\angle$; that is, $KB \perp AB$ at $B$.

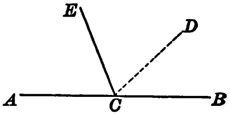
Prop. XXVII, page 114.

**Proof.** Const., $C$ and $F$ are each equidistant from $D$ and $E$; hence, § 106, $FC \perp DE$; that is, $CG \perp AB$.

Ex. **222.** Two angles of a triangle being given to construct the third.

**Solution.** Draw the straight line $AB$, and at any pt. in it, as $C$, § 241, construct $\angle$s $BCD$ and $DCE =$ the given $\angle$s respectively.

Then, $\angle ACE$ is the $\angle$ required.



**Proof.** Since, § 110, the sum of the $\angle$s of a $\triangle = 2$ rt. $\angle$s, and, § 56, the sum of the three $\angle$s $ACE$, $DCE$, and $BCD = 2$ rt. $\angle$s, and since, const., two of the $\angle$s of the $\triangle = \angle$s $DCE$ and $BCD$, resp., Ax. 3, the third $\angle$ of the $\triangle = \angle ACE$.

Ex. **223.** Two secants cut each other without a circle; the intercepted arcs are 72° and 48°. What is the angle between the secants?

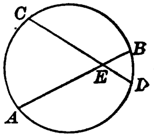
**Solution.** § 234, the required $\angle$ is measured by an arc of $\frac{1}{2}(72° - 48°)$, or 12°; consequently, the required $\angle = 12°$.

Ex. **224.** A tangent and a secant cut each other without a circle; the intercepted arcs are 94° and 32°. What is the angle between the tangent and the secant?

**Solution.** § 233, the required $\angle$ is measured by an arc of $\frac{1}{2}(94° - 32°)$, or 31°; consequently, the required $\angle = 31°$.

Ex. **225.** Two chords of a circle intersect and two opposite intercepted arcs are 88° and 26°. What are the angles between the chords?

**Solution.** Let the chords $AB$ and $CD$, intersecting at $E$, intercept the opposite arcs $AC$ and $BD$ equal respectively to 88° and 26°.



Then, § 230, $\angle AEC$ is measured by

$\frac{1}{2}(\text{arc } AC + \text{arc } BD)$, or $\frac{1}{2}(88° + 26°) = 57°$;

consequently, $\angle AEC = 57°$, and, § 59, $\angle BED = 57°$.

Also, § 55, $\angle BEC = 180° - 57° = 123°$; similarly, $\angle AED = 123°$.

Ex. **226.** A tangent of a circle and a chord from the point of contact intercept an arc of 110°. What is the angle between the tangent and the chord?

**Solution.** § 231, the required $\angle$ is measured by an arc of $\frac{1}{2}$ of 110°, or 55°; consequently, the required $\angle = 55°$.

Ex. **227.** If the radii of two intersecting circles are $4^{dm}$ and $9^{dm}$, respectively, what is the greatest and the least possible distance between their centers?

**Solution.** If the center of one circle were at the distance $9^{dm} - 4^{dm} = 5^{dm}$ from the center of the other, the circles would be tangent internally and would not intersect; therefore, the least line of centers would be just a little more than $5^{dm}$.

If their line of centers were $9^{dm} + 4^{dm} = 13^{dm}$, the circles would be tangent externally and would not intersect; therefore, their greatest line of centers would be just a little less than $13^{dm}$.

PROP. XXX, page 116.

**Proof.** Const., $AB = n$, $AC = m$, and $\angle A = \angle r$;

$\therefore$ § 100, $\triangle ABC =$ the $\triangle$ whose parts are given;

hence, $\triangle ABC$ is the $\triangle$ required.

PROP. XXXI, page 116.

**Proof.** Const., $AB = m$, $\angle A = \angle r$, and $\angle B = \angle s$;

$\therefore$ § 102, $\triangle ABC =$ the $\triangle$ whose parts are given;

hence, $\triangle ABC$ is the $\triangle$ required.

PROP. XXXII, page 117.

**Proof.** Const., $AB = o$, $AC = n$, and $BC = m$;

$\therefore$ § 107, $\triangle ABC =$ the $\triangle$ whose parts are given;

hence, $\triangle ABC$ is the $\triangle$ required.

Ex. **228**. Construct an equilateral triangle.

**Solution.** Assume any line as a side and proceed as in § 246.

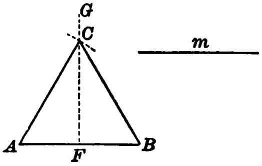
**Another Solution.** Let line $m$ be a side. Draw $AB=m$. Draw $FG \perp AB$ at its middle point. From $A$ as a center, with radius equal to $m$ describe an arc intersecting $FG$ as at $C$. Draw $AC$ and $BC$.

Then, $\triangle ABC$ is the $\triangle$ required.

**Proof.** Const., $AB = m$ and $AC = m$;

$\therefore AB = AC$, and since, § 103, $BC = AC$,

$\triangle ABC$ is equilateral.



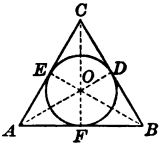
Ex. **229**. Prove that the radius of a circle inscribed in an equilateral triangle is equal to one third the altitude of the triangle.

**Data**: Equilateral $\triangle ABC$ and the inscribed circle whose center is $O$.

**To prove** the radius of the circle equal to one third the altitude of the $\triangle$.

**Proof.** Draw the radius $OF$ to the point of contact of $AB$ and draw $OA$ and $OB$.

Then, § 205, $OF \perp AB$, and in rt. $\triangle AFO$ and $BFO$, since, § 117, $\angle CAB = \angle ABC$, Ex. 221 and Ax. 7, $\angle OAF = \angle OBF$, and $OF$ is common; $\therefore$ § 114, $\triangle AFO = \triangle BFO$, and $AF = BF$; hence, § 121, $FO$ produced passes through $C$. Produce $FO$ to $C$.



Then, § 99, $COF$ is a median.

Similarly, lines $AD$ and $BE$ to the points of contact of $BC$ and $AC$ are medians and pass through $O$.

But, § 168, $CO = \frac{2}{3} CF$, or $OF = \frac{1}{3} CF$;

that is, the radius of the circle is equal to one third the altitude of the $\triangle$.

Ex. **230**. In an inscribed trapezoid the non-parallel sides are equal and the diagonals are equal.

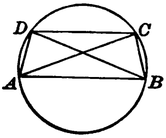
**Data**: Inscribed trapezoid $ABCD$, whose non-parallel sides are $AD$ and $BC$, and whose diagonals are $AC$ and $BD$.

**To prove** $AD = BC$, and $AC = BD$.

**Proof.** § 206, arc $AD$ = arc $BC$;

$\therefore$ § 196, $AD = BC$.

Hence, Ex. 138, $AC = BD$.



Prop. XXXIII, page 118.

**Proof.** Const., and § 148, $ABDC$ is a parallelogram, and by § 155, it is equal to the parallelogram whose parts are given; hence, $ABDC$ is the parallelogram required.

Prop. XXXV, page 119.

**Proof.** By construction parallel lines intercept equal parts on the transversal $AC$; hence, § 157, they intercept equal parts on transversal $AB$.

Ex. **231**. If the sides of a central angle of 35° intercept an arc of $75^{cm}$, what will be the length of an arc intercepted by the sides of a central angle of 80° in the same circle?

**Solution**. Since a central $\angle$ of 35° intercepts an arc of $75^{cm}$, an angle of 80° intercepts an arc of $\frac{80}{35}$, or $\frac{16}{7}$ of $75^{cm}$, which is $171\frac{3}{7}^{cm}$.

Ex. **232**. $AB$ and $CD$ are diameters of the circle whose center is $O$; $BD$ is an arc of 116°. How many degrees are there in each angle of triangles $AOC$ and $DOB$?

**Solution**. Data, arc $BD = 116°$; $\therefore$ § 224, $\angle BOD = 116°$.

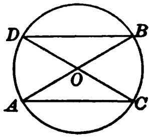
Ax. 14, $OD = OB$; $\therefore$ § 116, $\angle ODB = \angle OBD$,

and, § 110, $\angle ODB + \angle OBD = 180° - 116° = 64°$;

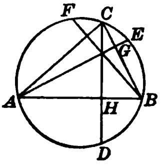
consequently, $\angle ODB = 32°$, and $\angle OBD = 32°$.

Since, § 59, $\angle AOC = \angle BOD = 116°$,

similarly, $\angle OAC = 32°$, and $\angle OCA = 32°$.



Ex. **233**. If a circle is circumscribed about a triangle $ABC$, and perpendiculars are drawn from the vertices to the opposite sides and produced to meet the circumference in the points $D$, $E$, and $F$, the arcs $EF$, $FD$, and $DE$ are bisected at the vertices.



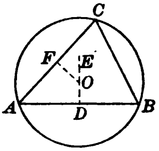
**Proof.** Let $AE$ cut $BC$ in $G$ and $CD$ cut $AB$ in $H$.
Then, in $\triangle s$ $ABG$ and $CBH$,
$\angle ABG$ is common, and, § 52, $\angle AGB = \angle CHB$;
∴ § 113, $\angle BAE = \angle BCD$;
but, § 225, $\angle BAE$ is measured by $\frac{1}{2}$ arc $BE$,
and $\angle BCD$ is measured by $\frac{1}{2}$ arc $BD$;
consequently, these measures are equal,
and, Ax. 6, arc $BE$ = arc $BD$.
Similarly, arc $CE$ = arc $CF$, and arc $AF$ = arc $AD$.

PROP. XXXVI, page 120.

**Proof.** By construction, the lines which intersect at $O$ are the perpendicular bisectors of two non-parallel chords; hence, § 201, $O$ is the center of the circle.

Ex. **234.** To circumscribe a circle about a given triangle.

**Solution.** Draw the perpendicular bisectors, $DE$ and $FO$, of two sides $AB$ and $AC$ of the given $\triangle ABC$ and produce them to intersect, as at $O$.

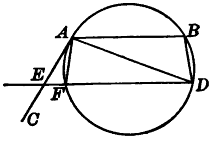


With $O$ as a center, and a radius $OA$, describe a circle.
Then, this circle circumscribes $\triangle ABC$.

**Proof.** § 103, $O$ is equidistant from $A$ and $B$,
and $O$ is equidistant from $A$ and $C$;
∴ $O$ is equidistant from $A$, $B$, and $C$,
and the circumference with $O$ as a center and radius $OA$ passes through the vertices $A$, $B$, and $C$.

Hence, § 189, this circle is circumscribed about $\triangle ABC$.

Ex. **235.** $AB$ is a chord of a circle and $AC$ is a tangent at $A$; a secant parallel to $AB$, as $EFD$, cuts $AC$ in $E$ and the circumference in $F$ and $D$; the lines $AF$, $AD$, and $BD$ are drawn. Prove that the triangles $AEF$ and $ADB$ are mutually equiangular.



**Proof.** § 231, $\angle EAF$ is measured by $\frac{1}{2}$ arc $AF$,
and, § 225, $\angle DAB$ is measured by $\frac{1}{2}$ arc $BD$;
but, § 206, arc $AF$ = arc $BD$;
∴ $\angle EAF = \angle DAB$.
§ 233, $\angle AEF$ is measured by $\frac{1}{2}$ (arc $ABD$ − arc $AF$);
but arc $AF$ = arc $BD$;
∴ $\angle AEF$ is measured by $\frac{1}{2}$ (arc $ABD$ − arc $BD$), or $\frac{1}{2}$ arc $AB$,
§ 225, $\angle ADB$ is measured by $\frac{1}{2}$ arc $AB$;
∴ $\angle AEF = \angle ADB$.
Hence, § 113, $\angle EFA = \angle ABD$.
Consequently, $\triangle s$ $AEF$ and $ADB$ are mutually equiangular.

PROP. XXXVII, page 120.

**Proof.** **Case I.** Const., $EF$ is perpendicular to the radius $OA$ at its extremity $A$; hence, § 205, $EF$ is tangent to the circle at $A$.

**Case II.** Draw the radii $OC$ and $OD$.

Then, § 227, $\triangle ACO$ and $ADO$ are rt. $\triangle$; that is, $AC$ and $AD$ are each perpendicular to a radius at its extremity; hence, § 205, $AC$ and $AD$ are tangents to the circle through $A$.

Ex. **236.** $ABC$ is an inscribed isosceles triangle; the vertical angle $C$ is three times each base angle. How many degrees are there in each of the arcs $AB$, $AC$, and $BC$?

**Solution.** Data, $\angle B = \angle A$, and $\angle C = 3\angle A$;

hence, § 110, $\angle A + \angle B + \angle C = \angle A + \angle A + 3\angle A = 5\angle A = 180°$,

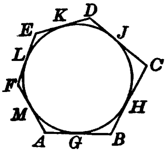
and $\angle A = 36°$; $\therefore \angle B = 36°$, and $\angle C = 108°$.

§ 225, $\angle A$ is measured by $\frac{1}{2}$ arc $BC$;

hence, $\frac{1}{2}$ arc $BC = 36°$, and arc $BC = 72°$;

similarly, arc $AC = 72°$, and arc $AB = 216°$.

Ex. **237.** If a hexagon is circumscribed about a circle, the sums of its alternate sides are equal.



**Datum**: Circumscribed hexagon $ABCDEF$.

**To prove** $AB + CD + EF = BC + DE + AF$.

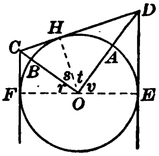
**Proof.** Denote the points of tangency of the sides by $G$, $H$, $J$, $K$, $L$, $M$.

Then, § 209, $AG = AM$, $BG = BH$, $CJ = CH$,

$DJ = DK$, $EL = EK$, $FL = FM$;

hence, Ax. 2, $AB + CD + EF = BC + DE + AF$.

Ex. **238.** Two radii of a circle at right angles to each other are intersected, when produced, by a line tangent to the circle. Prove that the tangents drawn to the circle from the points of intersection are parallel to each other.



**Data**: The circle whose center is $O$; two radii $OA$ and $OB$ at rt. $\triangle$ to each other; a line tangent at $H$ and intersecting $OB$ and $OA$ produced in $C$ and $D$; $CF$ and $DE$ tangent at the points $F$ and $E$, respectively.

**To prove** $CF \parallel DE$.

**Proof.** Draw $OF$, $OE$, and $OH$.

Ex. 221, $\angle r = \angle s$, and $\angle v = \angle t$;

data and Ax. 9, $\angle s + \angle t = 1$ rt. $\angle$; $\therefore \angle r + \angle v = 1$ rt. $\angle$,

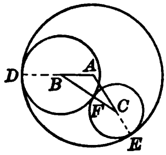
and $\angle r + \angle s + \angle t + \angle v = 2$ rt. $\triangle$, or, Ax. 9, $\angle FOD + \angle DOE = 2$ rt. $\triangle$;

hence, § 58, $FOE$ is a straight line;

consequently, § 176, $FOE$ is a diameter.

Hence, Ex. 196, $CF \parallel DE$.

**Ex. 239.** Two circles are tangent to each other externally and each is tangent to a third circle internally. Show that the perimeter of the triangle formed by joining the three centers is equal to the diameter of the exterior circle.



**Data**: Circles whose centers are $B$ and $C$, tangent to each other externally at $F$ and internally tangent to the circle whose center is $A$ at $D$ and $E$, respectively; also $\triangle ABC$.

**To prove** the perimeter of $\triangle ABC =$ diameter of the circle whose center is $A$.

**Proof.** Draw $BD$ and $CE$. $BD$ is in the same straight line with $AD$, for each would be $\perp$ to the common tangent at $D$, and, § 51, only one perpendicular can be drawn to a line at a point in it.

Similarly, $CE$ is in the same straight line with $AE$.

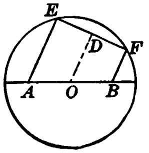
Ax. 14, $BF = BD$, and $CF = CE$;

hence, $AB + BF + CF + AC = AB + BD + AC + CE$,

or $AB + BC + AC = AD + AE$;

that is, perimeter of $\triangle ABC =$ diameter of circle whose center is $A$.

**Ex. 240.** From two points, $A$ and $B$, in a diameter of a circle, each the same distance from the center, two parallel lines $AE$ and $BF$ are drawn toward the same semicircumference, meeting it in $E$ and $F$. Show that $EF$ is perpendicular to $AE$ and $BF$.



**Proof.** Draw $OD$ from the center $O$ parallel to $AE$ and meeting $EF$ in $D$.

Then, § 80, $OD \parallel BF$.

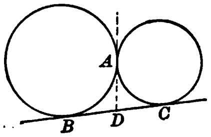
Since, data, $AO = BO$, § 157, $ED = FD$, and $D$ is the middle point of chord $EF$.

Now $O$ and $D$ are each equidistant from $E$ and $F$

$\therefore$ § 106, $OD \perp EF$;

hence, § 72, $EF \perp AE$, and $EF \perp BF$.

**Ex. 241.** Two circles are tangent externally at $A$. $BC$ is a tangent to the two circles at $B$ and $C$. Prove that the circumference of the circle described on $BC$ as a diameter passes through $A$.

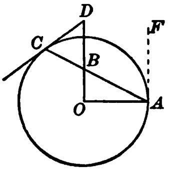


**Proof.** Through $A$ draw the common interior tangent $AD$ meeting $BC$ at $D$.

Then, § 209, $DB = DA = DC$.

Hence, a circumference described with $D$ as a center and a radius $DB$ passes through the points $B$, $A$, and $C$, and $BC$ is its diameter.

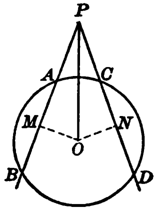
Ex. **242.** $OA$ is a radius of a circle whose center is $O$; $B$ is a point on a radius perpendicular to $OA$; through $B$ the chord $AC$ is drawn; at $C$ a tangent is drawn meeting $OB$ produced in $D$. Prove that $CBD$ is an isosceles triangle.



**Proof.** Through $A$ draw tangent $AF$.

§ 231, $\angle DCB$ is measured by $\frac{1}{2}$ arc $AC$,
and $\angle FAC$ is measured by $\frac{1}{2}$ arc $AC$;
$\therefore$ $\angle DCB = \angle FAC$;
but, § 76, $\angle CBD = \angle FAC$;
$\therefore$ $\angle DCB = \angle CBD$;
hence, § 118, $\triangle CBD$ is isosceles.

Ex. **243.** Through a given point $P$ without a circle whose center is $O$ two lines $PAB$ and $PCD$ are drawn, making equal angles with $OP$ and intersecting the circumference in $A$ and $B$, $C$ and $D$, respectively. Prove (1) that $AB$ equals $CD$, and (2) that $AP$ equals $CP$.

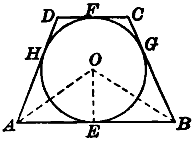


**Proof.** (1) Draw $OM$ and $ON$ perpendicular to $AB$ and $CD$, respectively.

Since $O$ is in the bisector of $\angle P$,
§ 134, $OM = ON$, and, § 202, $AB = CD$.

(2) In rt. $\triangle$ $PMO$ and $PNO$, $OP$ is common,
and, data, $\angle MPO = \angle NPO$;
$\therefore$ § 114, $\triangle PMO = \triangle PNO$,
and $PM = PN$; but, § 200 and Ax. 7, $AM = CN$;
hence, Ax. 3, $PM - AM = PN - CN$; that is, $AP = CP$.

Ex. **244.** If the angles at the base of a circumscribed trapezoid are equal, each non-parallel side is equal to half the sum of the parallel sides.



**Data:** Circumscribed trapezoid $ABCD$, whose sides are tangent at points $E$, $F$, $G$, and $H$, and whose base $\angle$ $A$ and $B$ are equal.

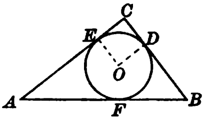
**To prove** $BC$ and $AD$ each equal to $\frac{1}{2}(AB+DC)$.

**Proof.** Draw $OA$, $OB$, and $OE$ from $O$ the center of the circle.

Then, § 205, $OE \perp AB$, and since, Ex. 221, $OA$ and $OB$ bisect $\angle$ $A$ and $B$, data and Ax. 7, $\angle OAB = \angle ABO$; hence, § 116, $\triangle ABO$ is isosceles, and, § 122, $AE = EB$, or $EB = \frac{1}{2}AB$.

Similarly, $FC = \frac{1}{2}DC$.
Now, § 209, $BG = EB$, and $CG = FC$;
$\therefore$ Ax. 2, $BG + CG = EB + FC$;
$\therefore$ substituting, $BG + CG = \frac{1}{2}AB + \frac{1}{2}DC$, or $BC = \frac{1}{2}(AB + DC)$.
Similarly, $AD = \frac{1}{2}(AB + DC)$.

Ex. **245**. If a circle is inscribed in a right triangle, the sum of the diameter and the hypotenuse is equal to the sum of the other two sides of the triangle.



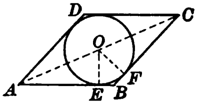
**Data:** The circle whose center is $O$ inscribed in rt. $\triangle ABC$ whose hypotenuse is $AB$.

**To prove** diameter of circle $+ AB = AC + BC$.

**Proof.** Draw radii $OE$ and $OD$ to the points of tangency of $AC$ and $BC$; let $F$ be the point of tangency of $AB$.

Then, § 205, $OE \perp AC$, and $OD \perp BC$;
hence, since $\angle C$ is a rt. $\angle$, § 71, $OE \parallel BC$, and $OD \parallel AC$;
$\therefore$ § 151, $OD = EC$, and $OE = DC$;
also, § 209, $AF = AE$, and $BF = BD$;
$\therefore$ Ax. 2, $(OD + OE) + (AF + BF) = (AE + EC) + (BD + DC)$;
that is, diameter of circle $+ AB = AC + BC$.

Ex. **246**. Any parallelogram which can be circumscribed about a circle is equilateral.



**Data:** Any parallelogram, as $ABCD$, circumscribed about the circle whose center is $O$.

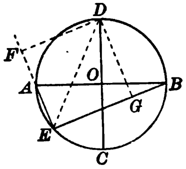
**To prove** $ABCD$ equilateral.

**Proof.** Draw $OA$ and $OC$; also draw radii $OE$ and $OF$ to the points of contact of $AB$ and $BC$, respectively.

§ 149, $\angle BAD = \angle BCD$; but, Ex. 221, $OA$ and $OC$ bisect these $\angle$s; hence, Ax. 7, $\angle OAE = \angle OCF$, Ax. 14, $OE = OF$, and, § 205, $\angle$s $AEO$ and $OFC$ are rt. $\angle$s; $\therefore$ § 114, $\triangle AEO = \triangle OFC$, and $AE = FC$. Also, § 209, $EB = BF$;
hence, Ax. 2, $AE + EB = BF + FC$, or $AB = BC$.
Now, § 153, $DC = AB$, and $AD = BC$;
hence, $DC = AD = AB = BC$,
and $ABCD$ is equilateral.

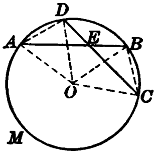
Ex. **247**. $AB$ and $CD$ are perpendicular diameters of a circle; $E$ is any point on the arc $ACB$. Then, $D$ is equidistant from $AE$ and $BE$.

**Proof.** Draw $DF \perp AE$, $DG \perp BE$, and draw $ED$.



Then, in rt. $\triangle$s $EFD$ and $EGD$, $ED$ is common,
§ 225, $\angle FED$ is measured by $\frac{1}{2}$ arc $AD$,
and $\angle DEG$ is measured by $\frac{1}{2}$ arc $BD$;
but, data and § 224, arc $AD =$ arc $BD$;
$\therefore$ $\angle FED = \angle DEG$;
hence, § 114, $\triangle EFD = \triangle EGD$, and $DF = DG$;
that is, $D$ is equidistant from $AE$ and $BE$.

**Ex. 248.** If two equal chords of a circle intersect, their corresponding segments are equal.



**Data**: Equal chords $AB$ and $CD$ intersecting at $E$.

**To prove** $AE = CE$, and $BE = DE$.

**Proof.** Draw $OA$, $OB$, $OD$, $OC$, $AD$, and $BC$.

§ 196, arc $AB$ = arc $DC$; ∴ § 194, $\angle AOB = \angle DOC$;

hence, Ax. 3, $\angle AOB - \angle DOB = \angle DOC - \angle DOB$,

or $\angle AOD = \angle BOC$;

also, Ax. 14, $AO = BO$, and $DO = CO$;

∴ § 100, $\triangle AOD = \triangle BOC$, and $AD = BC$.

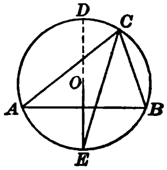
Then, in ▵ $AED$ and $BEC$, $AD = BC$,

§ 225, $\angle DAE = \angle BCE$ (each being measured by $\frac{1}{2}$ arc $DB$),

and $\angle ADE = \angle CBE$ (each being measured by $\frac{1}{2}$ arc $AMC$);

∴ § 102, $\triangle AED = \triangle BEC$, $AE = CE$, and $BE = DE$.

**Ex. 249.** If the arc cut off by the base of an inscribed triangle is bisected and from the point of bisection a radius is drawn and also a line to the opposite vertex, the angle between these lines is equal to half the difference of the angles at the base of the triangle.



**Data**: Inscribed $\triangle ABC$, in which $\angle B$ is greater than $\angle A$; $E$ the middle point of arc $AB$; the radius $OE$; and line $CE$.

**To prove** $\angle E = \frac{1}{2}(\angle B - \angle A)$.

**Proof.** Produce $EO$ to meet the circumference at $D$.

§ 192, arc $DAE$ = arc $DBE$,

but, data, arc $AE$ = arc $BE$;

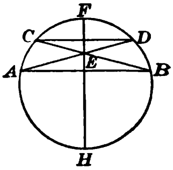
∴ Ax. 3, arc $AD$ = arc $BD$;

then, arc $AC$ – arc $BC$ = 2 arc $DC$;

hence, § 225, $\angle B - \angle A = 2\angle E$;

that is, $\angle E = \frac{1}{2}(\angle B - \angle A)$.

**Ex. 250.** The two lines which join the opposite extremities of two parallel chords intersect at a point in that diameter which is perpendicular to the chords.



**Data**: Two parallel chords, $AB$ and $CD$, and the diameter $FH$ perpendicular to them; also the lines $AD$ and $BC$ intersecting at $E$.

**To prove** $E$ lies in $FH$.

**Proof.** § 206, arc $AC$ = arc $BD$;

∴ § 225, $\angle CDE = \angle DCE$,

and, § 118, $CE = DE$;

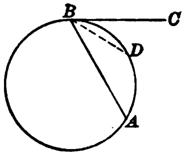
that is, $E$ is equidistant from $C$ and $D$;

hence, § 104, $E$ lies in the perpendicular to $CD$ at its middle point;

but, data and § 200, $FH \perp CD$ at its middle point;

therefore, $E$ lies in $FH$.

Ex. **251**. If a tangent is drawn to a circle at the extremity of a chord, the middle point of the subtended arc is equidistant from the chord and the tangent.



**Data**: Chord $AB$ and the tangent $BC$ at $B$; also $D$ the middle point of arc $AB$.

**To prove** $D$ equidistant from $AB$ and $BC$.

**Proof.** Draw $BD$.

§ 225, $\angle ABD$ is measured by $\frac{1}{2}$ arc $AD$,

and, § 231, $\angle DBC$ is measured by $\frac{1}{2}$ arc $BD$;

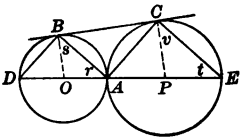
but, data, arc $AD =$ arc $BD$;

$\therefore$ $\angle ABD = \angle DBC$,

and $BD$ is the bisector of $\angle ABC$;

hence, § 134, $D$ is equidistant from $AB$ and $BC$.

Ex. **252**. A common exterior tangent to two circles is drawn. Prove that the chords, that join the points of contact with the points in which the line through the centers meets the circumferences, are parallel in pairs.



**Data** : Two circles whose centers are $O$ and $P$ tangent at $A$; their common exterior tangent at points $B$ and $C$; the line through $O$ and $P$ cutting the circumferences in $D$, $A$, and $E$; and the lines $DB$, $AB$, $AC$, and $EC$.

**To prove** $AB \parallel EC$, and $DB \parallel AC$.

**Proof.** Draw $OB$ and $PC$.

Then, § 205, $OB \perp BC$, and $PC \perp BC$; $\therefore$ § 71, $OB \parallel PC$;

hence, § 76, $\angle DOB = \angle APC$;

but, § 115, $\angle DOB = \angle r + \angle s$, and $\angle APC = \angle t + \angle v$;

hence, Ax. 1, $\angle r + \angle s = \angle t + \angle v$; but, Ax. 14, $OA = OB$;

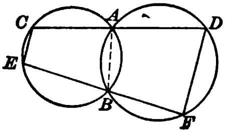
$\therefore$ § 116, $\angle r = \angle s$; similarly, $\angle t = \angle v$;

consequently, $2\angle r = 2\angle t$, and $\angle r = \angle t$;

$\therefore$ § 77, $AB \parallel EC$; similarly, $DB \parallel AC$.

Ex. **253**. Two circles intersect at the points $A$ and $B$; through $A$ a secant is drawn intersecting one circumference in $C$ and the other in $D$; through $B$ a secant is drawn intersecting the circumference $CAB$ in $E$ and the other circumference in $F$. Prove that the chords $CE$ and $DF$ are parallel.

**Proof.** Draw $AB$.



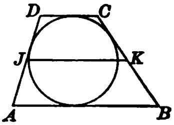
Ex. 198, $\angle CEB + \angle CAB = 2$ rt. $\angle$s;

but, Ex. 201, $\angle CAB = \angle BFD$;

$\therefore$ substituting, $\angle CEB + \angle BFD = 2$ rt. $\angle$s;

hence, § 79, $CE \parallel DF$.

**Ex. 254.** The length of the straight line joining the middle points of the non-parallel sides of a circumscribed trapezoid is equal to one fourth the perimeter of the trapezoid.

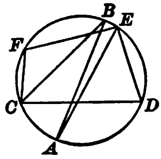


**Data:** Line $JK$ joining the middle points of the non-parallel sides of the circumscribed trapezoid $ABCD$.

**To prove** $JK = \frac{1}{4}$ perimeter of $ABCD$.

**Proof.** Ex. 213, $AB + DC = \frac{1}{2}$ perimeter of $ABCD$;
but, § 160, $JK = \frac{1}{2}(AB + DC)$;
hence, $JK = \frac{1}{4}$ perimeter of $ABCD$.

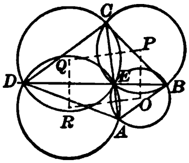
**Ex. 255.** A quadrilateral is inscribed in a circle, and two opposite angles are bisected by lines meeting the circumference in $A$ and $B$. Prove that $AB$ is a diameter.



**Proof.** Let $CDEF$ be the quadrilateral and the $\angle$s $C$ and $E$ the bisected $\angle$s.

Ex. 198, $\angle FCD + \angle FED = 2$ rt. $\angle$s;
∴ Ax. 7, $\angle FCB + \angle FEA = 1$ rt. $\angle = 90°$;
but, § 225, $\angle FCB$ is measured by $\frac{1}{2}$ arc $BF$,
and $\angle FEA$ is measured by $\frac{1}{2}$ arc $AF$;
∴ $\frac{1}{2}$ arc $BF + \frac{1}{2}$ arc $AF = 90°$,
and arc $BF +$ arc $AF =$ arc $AFB = 180°$;
consequently, $AB$ is a diameter.

**Ex. 256.** The centers of the four circles circumscribed about the four triangles formed by the sides and diagonals of a quadrilateral lie on the vertices of a parallelogram.



**Data:** Quadrilateral $ABCD$; its diagonals $AC$ and $DB$ intersecting in $E$; and the circles whose centers are $O$, $P$, $Q$, and $R$, circumscribing the $\triangle$s $ABE$, $BCE$, $DCE$, and $AED$.

**To prove** $O$, $P$, $Q$, and $R$ lie on the vertices of a ▱.

**Proof.** Join by lines the points $O$, $P$, $Q$, and $R$ in succession.

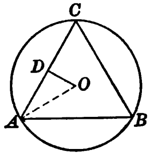
Since $O$ and $P$ are each equidistant from $E$ and $B$, § 106, $OP \perp EB$; similarly, $RQ \perp DE$; but $DEB$ is a straight line;
∴ § 71, $OP \parallel RQ$.

In like manner it may be shown that $OR \parallel PQ$.

Hence, § 140, $OPQR$ is a ▱;
that is, $O$, $P$, $Q$, and $R$ lie on the vertices of a ▱.

**Ex. 257.** If an equilateral triangle is inscribed in a circle, the distance of each side from the center is equal to half the radius of the circle.

**Data**: Inscribed equilateral $\triangle ABC$, and the perpendicular $OD$ representing the distance from the center to any side, as $AC$.



**To prove** $OD = \frac{1}{2}$ the radius.

**Proof.** Draw the radius $OA$.

Then, Ex. 208, $\angle AOD = \angle B$;

but, Ex. 74, $\angle B = 60°$; $\therefore \angle AOD = 60°$,

and, § 111, $\angle OAD = 90° - 60° = 30°$.

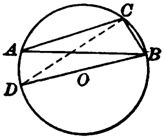
Hence, Ex. 172, $OD = \frac{1}{2} OA$;

that is, $OD = \frac{1}{2}$ the radius.

Ex. **258**. The vertical angle of an oblique triangle inscribed in a circle is greater or less than a right angle by the angle contained by the base and diameter drawn from the extremity of the base.

**Data**: Oblique $\triangle ABC$ inscribed in a circle whose center is $O$; its vertical $\angle C$; and the diameter $BD$ from one extremity of the base.

**Case I.** When $O$ lies without the $\triangle$.



**To prove** $\angle ACB >$ a rt. $\angle$ by $\angle ABD$.

**Proof.** Draw $DC$.

Then, Ax. 9, $\angle ACB = \angle DCB + \angle ACD$;

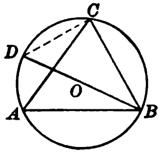
but, § 227, $\angle DCB =$ a rt. $\angle$,

and $\angle ACD = \angle ABD$ (each being measured by $\frac{1}{2}$ arc $AD$);

$\therefore$ $\angle ACB =$ a rt. $\angle + \angle ABD$;

that is, $\angle ACB >$ a rt. $\angle$ by $\angle ABD$.

**Case II.** When $O$ lies within the $\triangle$.



**To prove** $\angle ACB <$ a rt. $\angle$ by $\angle ABD$.

**Proof.** Draw $DC$.

§ 227, $\angle DCB =$ a rt. $\angle$;

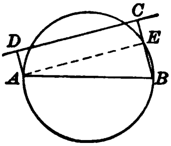
$\therefore$ Ax. 3, $\angle ACB =$ a rt. $\angle - \angle ACD$;

but $\angle ACD = \angle ABD$ (each being measured by $\frac{1}{2}$ arc $AD$);

hence, $\angle ACB =$ a rt. $\angle - \angle ABD$;

that is, $\angle ACB <$ a rt. $\angle$ by $\angle ABD$.

Ex. **259**. If from the extremities of any diameter of a given circle perpendiculars to any secant, that is not parallel to this diameter and does not intersect it, are drawn, the less perpendicular is equal to that segment of the greater which is contained between the circumference and the secant.



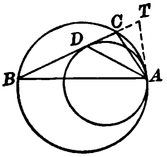
**Data**: Diameter $AB$ of a $\odot$, and the perpendiculars $AD$ and $BC$ from $A$ and $B$ respectively to secant $DC$ that is not parallel to, and does not intersect, $AB$, the longer perpendicular intersecting the circumference in $E$.

**To prove** $AD = EC$.

**Proof.** Draw $AE$.

§ 227, $\angle AEB$ is a rt. $\angle$, and, § 26, $AE \perp BC$,
also, data, $DC \perp BC$; $\therefore$ § 71, $AE \parallel DC$;
also, data and § 71, $AD \parallel EC$; hence, § 151, $AD = EC$.

**Ex. 260.** Two circles are tangent internally at $A$, and a chord $BC$ of the larger circle is tangent to the smaller at $D$. Prove that $AD$ bisects the angle $CAB$.

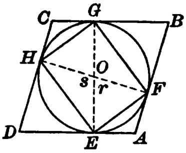


**Proof.** Draw the common tangent $AT$ to meet $BC$ produced as at $T$.

§ 209, $TD = TA$; $\therefore$ § 116, $\angle TDA = \angle TAD$;
but, § 115, $\angle TDA = \angle ABC + \angle DAB$,
and, Ax. 9, $\angle TAD = \angle TAC + \angle CAD$;
$\therefore$ Ax. 1, $\angle ABC + \angle DAB = \angle TAC + \angle CAD$;
but $\angle ABC = \angle TAC$ (each being measured by $\frac{1}{2}$ arc $AC$);
hence, Ax. 3, $\angle DAB = \angle CAD$;
that is, $AD$ bisects $\angle CAB$.

**Ex. 261.** The tangents at the four vertices of an inscribed rectangle form a rhombus.

**Data:** Inscribed rectangle $EFGH$, in which $EH > EF$, and the tangents at its vertices forming the figure $ABCD$.



**To prove** $ABCD$ a rhombus.

**Proof.** Draw the diagonals $EG$ and $FH$.

Since, § 141, $\angle EHG$ is a rt. $\angle$, and, § 225, is measured by $\frac{1}{2}$ arc $EFG$, this arc is 180°; consequently, $EG$ is a diameter, and, Ex. 196, $AD \parallel BC$; and similarly, $AB \parallel DC$; hence, §§ 190, 140, $ABCD$ is a circumscribed ▱.

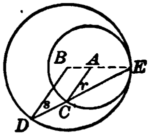
In $\triangle$s $EOH$ and $FOE$, Ax. 14, $HO = EO = FO$; but, data, $EH > EF$; $\therefore$ § 130, $\angle s$ is greater than $\angle r$; consequently, these angles are oblique.

In quadrilateral $EOHD$, $\angle$s $DEO$ and $OHD$ are rt. $\angle$s; hence, $\angle D$ is sup. of $\angle s$ and therefore is an oblique $\angle$.

Similarly, $\angle$s $A$, $B$, and $C$ are oblique $\angle$s.

Hence, § 142, $ABCD$ is a rhomboid, and since, Ex. 246, it is equilateral, $ABCD$ is a rhombus.

**Ex. 262.** If a line is drawn through the point of contact of two circles which are tangent internally, intersecting the circle whose center is $A$ at $C$, and the circle whose center is $B$ at $D$, $AC$ and $BD$ are parallel.

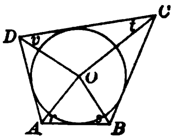


**Proof.** Let $E$ be the point of contact.

Draw the line of centers and, § 213, produce it to $E$.

Ax. 14, $AC = AE$; $\therefore$ § 116, $\angle r = \angle E$;
similarly, $\angle E = \angle s$; $\therefore$ $\angle r = \angle s$, and, § 77, $AC \parallel BD$.

**Ex. 263.** If lines are drawn from the center of a circle to the vertices of any circumscribed quadrilateral, each angle at the center is the supplement of the central angle that is not adjacent to it.

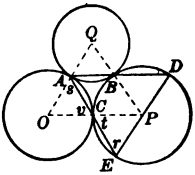


**Data:** Quadrilateral $ABCD$ circumscribed about the circle whose center is $O$, and the lines $OA$, $OB$, $OC$, and $OD$.

**To prove** $\angle$s $AOB$ and $DOC$ supplementary,
and $\angle$s $BOC$ and $AOD$ supplementary.

**Proof.** § 166, $\angle A + \angle B + \angle C + \angle D = 4$ rt. $\angle$s;
$\therefore$ Ex. 221 and Ax. 7, $\angle r + \angle s + \angle t + \angle v = 2$ rt. $\angle$s,
and $\angle r + \angle s$ is sup. of $\angle t + \angle v$;
but, § 110, $\angle AOB$ is sup. of $\angle r + \angle s$,
and $\angle DOC$ is sup. of $\angle t + \angle v$;
hence, $\angle$s $AOB$ and $DOC$ are supplementary.
Similarly, $\angle$s $BOC$ and $AOD$ are supplementary.

**Ex. 264.** Three circles are tangent to each other externally at the points $A$, $B$, and $C$. From $A$ lines are drawn through $B$ and $C$ meeting the circumference which passes through $B$ and $C$ at the points $D$ and $E$. Prove that $DE$ is a diameter.

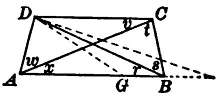


**Proof.** Draw the lines of centers $OP$, $OQ$, and $PQ$.

Ax. 14, $OA = OC$; $\therefore$ § 116, $\angle s = \angle v$, and since $PE = PC$, $\angle r = \angle t$; but, § 59, $\angle t = \angle v$; $\therefore$ $\angle r = \angle s$, and, § 75, $EP \parallel OQ$.

In like manner, $PD \parallel OQ$.
Hence, § 70, $EPD$ is a straight line,
and, § 176, $DE$ is a diameter.

**Ex. 265.** If an angle between a diagonal and one side of a quadrilateral is equal to the angle between the other diagonal and the opposite side, the same will be true of the three other pairs of angles corresponding to the same description, and the four vertices of the quadrilateral lie on a circumference.



**Data:** Quad. $ABCD$; its diagonals $AC$ and $BD$, and $\angle r = \angle v$.

**To prove** $A$, $B$, $C$, and $D$ lie on a circumference, $\angle s = \angle w$, $\angle t = \angle ADB$, and $\angle BDC = \angle x$.

**Proof.** § 207, a circumference may be passed through $A$, $C$, and $D$, and it will cut $AB$, or $AB$ produced as at $G$.

Then, $\angle AGD = \angle v$ (each being measured by $\frac{1}{2}$ arc $AD$);
but, data, $\angle ABD = \angle v$; $\therefore$ $\angle AGD = \angle ABD$;
hence, $G$ coincides with $B$, for if $G$ is at the left of $B$, § 115, $\angle AGD = \angle ABD + \angle GDB$, which is untenable for $\angle AGD = ABD$.

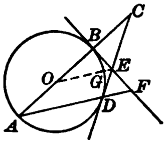
Similarly, if $G$ is on $AB$ produced, $\angle AGD$ would not be equal to $\angle ABD$.

Hence, $G$ and $B$ must coincide; that is, the circumference passes through $B$.

Then, $ABCD$ may be inscribed in a circle, and, § 225, $\angle s = \angle w$ (each being measured by $\frac{1}{2}$ arc $DC$).

Similarly, $\angle t = \angle ADB$, and $\angle BDC = \angle x$.

Ex. **266**. Let the diameter $AB$ of a circle be produced to $C$, making $BC$ equal to the radius; through $B$ draw a tangent, and from $C$ draw a second tangent to the circle at $D$, intersecting the first at $E$; $AD$ and $BE$ produced meet at $F$. Prove that $DEF$ is an equilateral triangle.



**Proof.** Let $O$ be the center of the circle, and draw $OE$, cutting the circumference in $G$.

§ 224, $\angle BOG$ is measured by arc $BG$,

and, § 225, $\angle BAD$ is measured by $\frac{1}{2}$ arc $BD$;

but, Ex. 221, arc $BG = \frac{1}{2}$ arc $BD$;

hence, $\angle BOG = \angle BAD$, and, § 77, $OE \parallel AF$.

Then, § 73, $\angle EDF = \angle OED$, and, § 76, $\angle DFE = \angle OEB$;

but, Ex. 221, $\angle OED = \angle OEB$; $\therefore \angle EDF = \angle DFE$.

Data, § 205, $EB \perp OC$ at its middle point;

$\therefore$ § 103, $OE = CE$, and, § 121, $\angle OEB = \angle CEB$;

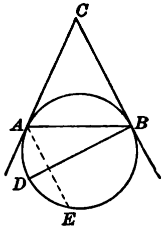
but, § 59, $\angle DEF = \angle CEB$; $\therefore \angle DEF = \angle OEB$,

and, since $\angle DFE = \angle OEB$, $\angle DEF = \angle DFE$;

consequently, $\angle EDF = \angle DFE = \angle DEF$.

Hence, § 119, $\triangle DEF$ is equilateral.

Ex. **267**. If from any point without a circle tangents are drawn, the angle contained by the tangents is double the angle contained by the line joining the points of contact and the diameter drawn through one of them.



**Data**: Tangents $AC$ and $BC$ from the point $C$ without a circle; line $AB$ joining the points of tangency; and the diameter $BD$.

**To prove** $\angle ACB = 2\angle ABD$.

**Proof.** Draw $AE \parallel CB$, meeting the circumference as at $E$.

§ 235, $\angle ACB$ is measured by $\frac{1}{2}$(arc $AEB$ − arc $AB$);

but, § 206, arc $AB$ = arc $BE$;

$\therefore$ arc $AEB$ − arc $AB$ = arc $AEB$ − arc $BE$ = arc $AE$;

consequently, $\angle ACB$ is measured by $\frac{1}{2}$ arc $AE$.

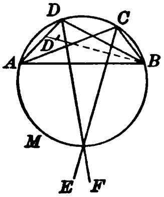
§ 225, $\angle ABD$ is measured by $\frac{1}{2}$ arc $AD$;

but since, § 205, $DB \perp BC$, § 72, $DB \perp AE$, and, § 200, arc $AD = \frac{1}{2}$ arc $AE$;

hence, $\angle ABD$ is measured by $\frac{1}{2}$ of $\frac{1}{2}$ arc $AE$, or $\frac{1}{4}$ arc $AE$.

Then, since the measure of $\angle ACB$ is double the measure of $\angle ABD$, $\angle ACB = 2\angle ABD$.

Ex. **268.** The lines which bisect the vertical angles of all triangles on the same base and on the same side of it, and having equal vertical angles, meet at the same point.



**Data:** Any $\triangle$s, as $ABC$ and $ABD$, on the base $AB$, and having their vertical $\angle$s $ACB$ and $ADB$ equal; also the bisectors, $CE$ and $DF$, of these $\angle$s respectively.

**To prove** $CE$ and $DF$ meet at the same point.

**Proof.** § 207, a circumference may be passed through $A$, $B$, and $C$, and if it does not pass through $D$, it will cut $AD$, or $AD$ produced, as at $D'$. Draw $BD'$.

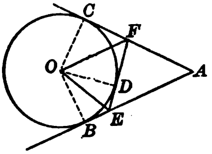
§ 225, $\angle AD'B = \angle ACB$ (each being measured by $\frac{1}{2}$ arc $AMB$),
but, data, $\angle ADB = \angle ACB$; $\therefore\ \angle AD'B = \angle ADB$.

Now, § 115, $\angle AD'B = \angle ADB + \angle DBD'$;
$\therefore\ \angle ADB = \angle ADB + \angle DBD'$, which is absurd, and the supposition that the circumference passes through $D'$, and not through $D$, is untenable; hence, $D$ is on the circumference.

Now, since, § 225, $\angle$s $ACB$ and $ADB$ are each measured by $\frac{1}{2}$ arc $AMB$, their bisectors must pass through the middle point of arc $AMB$; that is, $CE$ and $CF$ meet in the same point.

**Cor.** *All triangles on the same base, and with equal vertical angles, have their vertices on an arc of a circle, of which the given base is a chord.*

Ex. **269.** $AB$ and $AC$ are tangents at $B$ and $C$, respectively, to a circle whose center is $O$; from $D$, any point on the circumference, a tangent is drawn, meeting $AB$ in $E$, and $AC$ in $F$.

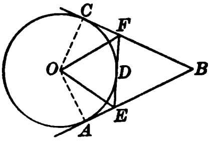


Prove that angle $EOF$ is equal to one half angle $BOC$.

**Proof.** Draw $OD$.

Ex. 221, $\angle DOE = \frac{1}{2}\angle BOD$,
and $\angle DOF = \frac{1}{2}\angle DOC$;
$\therefore$ Ax. 2, $\angle DOE + DOF = \frac{1}{2}(\angle BOD + \angle DOC)$.
or, Ax. 9, $\angle EOF = \frac{1}{2}\angle BOC$.

Ex. **270.** A circle whose center is $O$ is tangent to the sides of an angle $ABC$ at $A$ and $C$; through any point in the arc $AC$, as $D$, a tangent is drawn, meeting $AB$ in $E$, and $CB$ in $F$. Prove (1) that the perimeter of the triangle $BEF$ is constant for all positions of $D$ in $AC$; (2) that the angle $EOF$ is constant.



**Proof.** (1) Draw $OA$ and $OC$.
§ 209, $DE = EA$, and $FD = FC$;
$\therefore$ Ax. 2, $DE + FD = EA + FC$,
and adding $BE + BF$ to each member, Ax. 2,

$$BE + BF + (DE + FD) = (BE + EA) + (BF + FC),$$

or perimeter of $\triangle BEF = AB + BC$;
but $AB + BC$ is constant for all positions of $D$ in $AC$;

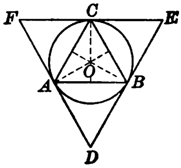
hence, perimeter of $\triangle BEF$ is constant.
(2) Ex. 269, $\angle EOF = \frac{1}{2}\angle AOC$;
but $\angle AOC$ is constant;
hence, $\angle EOF$ is constant.

Ex. **271**. If $AE$ and $BD$ are drawn perpendicular to the sides $BC$ and $AC$, respectively, of the triangle $ABC$, and $DE$ is drawn, the angles $AED$ and $ABD$ are equal.

**Proof.** § 207, pass a circumference through $A$, $D$, and $B$.

Then, since, data, $\angle$s $ADB$ and $AEB$ are rt. $\angle$s,
Ex. 268, Cor., this circumference passes through $E$.
Now, § 225, $\angle AED$ is measured by $\frac{1}{2}$ arc $AD$,
and $\angle ABD$ is measured by $\frac{1}{2}$ arc $AD$;
hence, $\angle AED = \angle ABD$.

Ex. **272**. The perimeter of an inscribed equilateral triangle is equal to one half the perimeter of the circumscribed equilateral triangle.



**Proof.** Let $\triangle ABC$ be an inscribed equilateral $\triangle$, and let the tangents at the vertices meet to form the circumscribed $\triangle DEF$.

Since, §§ 199, 121, the lines $\perp$ to the sides of $\triangle ABC$ at their middle points pass through the center of the $\odot$ and the opposite vertices, § 205, they are $\perp$ to the tangents through those vertices;
hence, § 71, $AB \parallel EF$, $BC \parallel DF$, and $AC \parallel DE$;
$\therefore$ § 153, $AC = DB$, and $AC = BE$;
hence, Ax. 2, $2\,AC = DB + BE = DE$;
similarly, $2\,AB = EF$, and $2\,BC = DF$;
consequently, Ax. 6, $DE = EF = DF$;
that is, $\triangle DEF$ is equilateral.
Also, Ax. 2, $2(AC + AB + BC) = DE + EF + DF$,
or $AC + AB + BC = \frac{1}{2}(DE + EF + DF)$;
that is, perimeter of $\triangle ABC = \frac{1}{2}$ perimeter of $\triangle DEF$.

Ex. **273**. In the circumscribed quadrilateral $ABCD$, the angles $A$, $B$, and $C$ are 110°, 95°, and 80°, respectively, and the sides $AB$, $BC$, $CD$, and $DA$ touch the circumference at the points $E$, $F$, $G$, and $H$, respectively. Find the number of degrees in each angle of the quadrilateral $EFGH$.

**Solution.** § 235, $\angle A$ is measured by $\frac{1}{2}$(arc $EFGH$ − arc $EH$);
hence, $\frac{1}{2}(\text{arc } EFGH - \text{arc } EH) = 110°$,
and $\text{arc } EFGH - \text{arc } EH = 220°$; (1)
but $\text{arc } EFGH + \text{arc } EH = 360°$; (2)

subtracting (1) from (2),

$2 \text{ arc } EH = 140°$, and arc $EH = 70°$.

Similarly, arc $EF = 85°$, and arc $FG = 100°$;

$\therefore$ arc $HG = 360° - (70° + 85° + 100°) = 105°$.

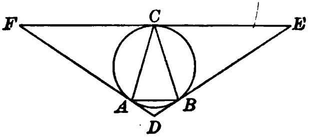
§ 225, $\angle HGF$ is measured by $\frac{1}{2}$ arc $HEF$, or $\frac{1}{2}(70° + 85°)$;

consequently, $\angle HGF = 77\frac{1}{2}°$,

and, Ex. 198, $\angle HEF = 180° - 77\frac{1}{2}° = 102\frac{1}{2}°$.

Similarly, $\angle EFG = 87\frac{1}{2}°$, and $\angle EHG = 180° - 87\frac{1}{2}° = 92\frac{1}{2}°$.

Ex. **274**. If an inscribed isosceles triangle has each of its base angles double the vertical angle, and its vertices the points of contact of three tangents, these tangents form an isosceles triangle, each of whose base angles is one third its vertical angle.



**Data**: Inscribed isosceles $\triangle ABC$, in which $\angle A = \angle B = 2\angle C$; and the tangents at $A$, $B$, and $C$ forming the $\triangle DEF$.

**To prove** $\triangle DEF$ isosceles, having $\angle$s $E$ and $F$, each one third of $\angle D$.

**Proof.** Data, $\angle A = \angle B = 2\angle C$;

$\therefore$ § 110, $\angle A + \angle B + \angle C = 2\angle C + 2\angle C + \angle C = 5\angle C = 180°$,

and $\angle C = 36°$;

hence, $\angle A = 72°$, and $\angle B = 72°$;

consequently, § 225, arc $AB = 72°$, arc $BC = 144°$, and arc $AC = 144°$.

§ 235, $\angle D$ is measured by $\frac{1}{2}(\text{arc } ACB - \text{arc } AB)$, or $\frac{1}{2}(288° - 72°)$;

hence, $\angle D = 108°$.

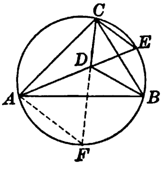
Similarly, $\angle E = 36°$, and $\angle F = 36°$.

Hence, § 118, $\triangle DEF$ is isosceles,

and since $36°$ is $\frac{1}{3}$ of $108°$,

$\angle$s $E$ and $F$ are each one third of $\angle D$.

Ex. **275**. $ABC$ is a triangle inscribed in a circle; the bisectors of the angles $A$, $B$, and $C$ meet in $D$; $AD$ produced meets the circumference in $E$. Prove that $DE$ equals $CE$.



**Proof.** Produce $CD$ to meet the circumference as at $F$ and draw $AF$.

Data, $\angle ACF = \angle FCB$, and $\angle CAE = \angle BAE$;

consequently, § 225, arc $AF$ = arc $FB$,

and arc $CE$ = arc $BE$.

§ 230, $\angle CDE$ is measured by $\frac{1}{2}$ (arc $AF$ + arc $CE$);

∴ substituting equal arcs,

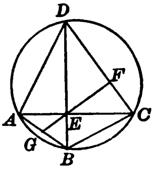
$\angle CDE$ is measured by $\frac{1}{2}$ (arc $FB$ + arc $BE$), or $\frac{1}{2}$ arc $FBE$,

but, § 225, $\angle FCE$ or $\angle DCE$ is measured by $\frac{1}{2}$ arc $FBE$;

∴ $\angle CDE = \angle DCE$;

hence, § 118, $DE = CE$.

**Ex. 276.** If the diagonals of a quadrilateral inscribed in a circle intersect at right angles, the perpendicular from their intersection upon any side, if produced, bisects the opposite side.



**Data:** Inscribed quadrilateral $ABCD$ whose diagonals $AC$ and $BD$ intersect at rt. $\angle$s in $E$; also $EF \perp DC$ and produced through $E$ to meet $AB$ at $G$.

**To prove** $AG = BG$.

**Proof.** § 115, $\angle CEG = \angle EFC + \angle FCE$;

but, data, § 52, $\angle BEC = \angle EFC$;

∴ Ax. 3, $\angle BEG = \angle FCE$;

but $\angle EBG = \angle FCE$ (each being measured by $\frac{1}{2}$ arc $AD$);

hence, Ax. 1, $\angle BEG = \angle EBG$;

and, § 118, $BG = GE$.

Similarly, $AG = GE$;

hence, $AG = BG$.

**Ex. 277.** If the opposite angles of a quadrilateral are supplementary, the quadrilateral may be inscribed in a circle.

**Data:** (See Fig. page 127 of Geom.) Quadrilateral $ABCD$ whose opposite angles are supplementary.

**To prove** $ABCD$ may be inscribed in a circle.

**Proof.** § 207, a circumference may be passed through $A$, $B$, and $D$, and if it does not pass through $C$ it will cut $DC$, or $DC$ produced, as at $E$. Draw $EB$.

Then, Ex. 198, $\angle DEB$ is sup. of $\angle A$;

but, data, $\angle DCB$ is sup. of $\angle A$;

∴ § 54, $\angle DCB = \angle DEB$.

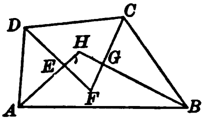
Now, § 115, $\angle DEB = \angle CBE + \angle DCB$;

∴ $\angle DCB = \angle CBE + \angle DCB$,

which is absurd, and the supposition that the circumference passes through $E$ and not through $C$ is untenable.

Hence, the circumference through $A$, $B$, and $D$ passes through $C$, and $ABCD$ may be inscribed in a circle.

**Ex. 278.** The four lines bisecting the angles of any quadrilateral form a quadrilateral which may be inscribed in a circle.



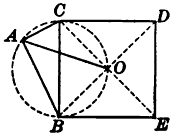
**Data:** Quadrilateral $ABCD$ and its angle bisectors meeting to form the quadrilateral $EFGH$.

**To prove** $EFGH$ may be inscribed in a circle.

**Proof.** By Ex. 179, the opposite $\angle$s of $EFGH$ are supplementary.

Hence, Ex. 277, $EFGH$ may be inscribed in a circle.

Ex. **279.** The line, drawn from the center of the square described upon the hypotenuse of a right triangle to the vertex of the right angle, bisects the right angle.



**Data:** Rt. $\triangle ABC$ and the square $BEDC$ described upon the hypotenuse $BC$; also the line $AO$ from the vertex of the rt. $\angle$ to $O$ the center of the square.

**To prove** $AO$ bisects $\angle BAC$.

**Proof.** Draw the diagonals $BD$ and $EC$.

Ex. 110, $\angle BOC$ is a rt. $\angle$, and, data, $\angle BAC$ is a rt. $\angle$;

consequently, $\angle$s $BOC$ and $BAC$ are supplementary,

and, § 166, $\angle$s $ABO$ and $ACO$ are supplementary;

$\therefore$ Ex. 277, $ABOC$ may be inscribed in a circle.

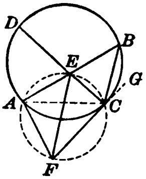
Ex. 112, $BD = EC$; $\therefore$ § 154 and Ax. 7, $BO = CO$,

and, § 196, arc $BO$ = arc $CO$;

hence, $\angle BAO = \angle CAO$ (being measured by the halves of equal arcs);

that is, $AO$ bisects $\angle BAC$.

Ex. **280.** $AB$ and $CD$ are two chords of a circle intersecting at $E$; through $A$ a line is drawn to meet a line tangent at $C$ so that angle $AFC$ equals the angle $BEC$. Then, $EF$ is parallel to $BC$.



**Proof.** Produce $FC$ to $G$, and draw $AC$.

§ 55, $\angle BEC$ is the supplement of $\angle AEC$;

but, data, $\angle AFC = \angle BEC$;

$\therefore$ $\angle AFC$ is the supplement of $\angle AEC$,

consequently, in quadrilateral $AFCE$, $\angle$s $FAE$ and $FCE$ are supplementary; hence, Ex. 277, $AFCE$ may be inscribed in a circle.

Then, § 225, $\angle EFC = \angle EAC$ (each being measured by $\frac{1}{2}$ arc $EC$),

and, §§ 231, 225, $\angle BCG = \angle EAC$ (each being measured by $\frac{1}{2}$ arc $BC$).

Hence, $\angle EFC = \angle BCG$, and, § 77, $EF \parallel BC$.

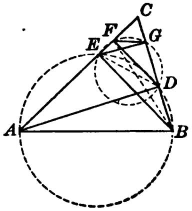
**Ex. 281.** $ABC$ is a triangle; $AD$ and $BE$ are the perpendiculars from $A$ and $B$ upon $BC$ and $AC$ respectively; $DF$ and $EG$ are perpendiculars from $D$ and $E$ upon $AC$ and $BC$ respectively. Then, $FG$ is parallel to $AB$.

**Proof.** Draw $ED$ and $BF$.

Data, § 52, $\angle EGD = \angle EFD$;

$\therefore$ Ex. 268, Cor., $EFGD$ may be inscribed in a circle.

Similarly, $ABDE$ may be inscribed in a circle.



Then, § 225, $\angle GFD = \angle GED$ (each being measured by $\frac{1}{2}$ arc $GD$), and $\angle ABE = \angle ADE$ (each being measured by $\frac{1}{2}$ arc $AE$); but, § 71, $EG \parallel AD$;

$\therefore$ § 73, $\angle GED = \angle ADE$; hence, $\angle GFD = \angle ABE$;

similarly, $\angle BFD = \angle FBE$;

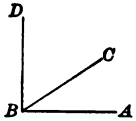
$\therefore$ Ax. 2, $\angle GFD + \angle BFD = \angle ABE + \angle FBE$;

that is, $\angle BFG = \angle ABF$; hence, § 75, $FG \parallel AB$.

**Ex. 282.** Construct the complement of a given angle.

**Solution.** § 239, draw $DB \perp AB$ at $B$, the vertex of the given $\angle ABC$.

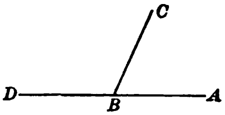
Then, § 31, $\angle CBD$ is the complement of $\angle ABC$.



**Ex. 283.** Construct the supplement of a given angle.

**Solution.** Produce one side, as $AB$, of the given $\angle ABC$ through the vertex to $D$.

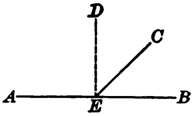
Then, §§ 55, 32, $\angle CBD$ is the sup. of $\angle ABC$.



**Ex. 284.** At a given point in a given straight line, to construct an angle of 45°.

**Solution.** At $E$, the given point, draw $ED \perp AB$, the given line, and, § 238, draw $EC$ to bisect $\angle BED$.

Then, $\angle BEC$ is an angle of 45°.

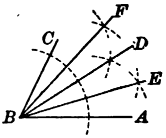


**Ex. 285.** Divide an angle into four equal parts.

**Solution.** Draw $BD$ to bisect the given $\angle ABC$.

Also bisect ∡ $ABD$ and $DBC$ by $BE$ and $BF$ respectively.

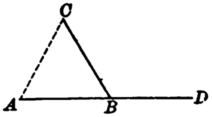
Then, $\angle ABC$ is divided into four equal parts.



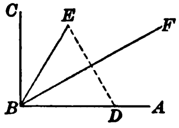
**Ex. 286.** At a given point in a given straight line, to construct an angle of 60°.

**Solution.** Ex. 228, construct an equilateral $\triangle$, as $ABC$, with one vertex at $B$, the given point in the given line $AD$.

Then, Ex. 74, $\angle ABC$ is an angle of 60°.



**Ex. 287.** Trisect a right angle.

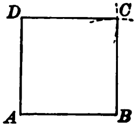


**Solution.** Ex. 286, at $B$, the vertex of the given rt. $\angle ABC$, construct $\angle ABE = 60°$, and draw $BF$ to bisect $\angle ABE$.

Then, $BF$ and $BE$ trisect $\angle ABC$.

**Ex. 288.** Construct a square, having given one side.

**Solution.** At one extremity $A$ of the given line $AB$ draw $AD \perp AB$ and equal to $AB$.



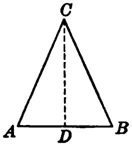
With $B$ and $D$ as centers and radii equal to $AB$ describe arcs intersecting as at $C$. Draw $DC$ and $BC$.

Then, $ABCD$ is the required square.

**Proof.** Const., the sides of $ABCD$ are each equal to $AB$; hence, § 148, $ABCD$ is a ▱, and since, const., $\angle BAD$ is a rt. $\angle$, Ex. 104, each $\angle$ of $ABCD$ is a rt. $\angle$.

Hence, § 143, $ABCD$ is a square.

**Ex. 289.** Construct an isosceles triangle, having given the base and the perpendicular from the vertex to the base.



**Solution.** Draw $AB =$ the given base, and at its middle point erect the perpendicular $DC =$ the given perpendicular. Draw $AC$ and $BC$.

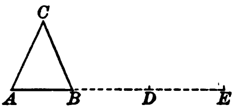
**Proof.** § 103, $AC = BC$, and, § 90, $\triangle ABC$ is isosceles.

**Ex. 290.** Construct an equilateral triangle, having given the perimeter.

**Solution.** § 250, divide the given perimeter into three equal parts and with these parts as sides, § 246, construct $\triangle ABC$.

Then, $\triangle ABC$ is the required $\triangle$.

**Ex. 291.** Construct an isosceles triangle, having given the perimeter and base.



**Solution.** On $AE =$ given perimeter measure $AB =$ given base, and find $D$ the middle pt. of $BE$.

§ 246, with base $AB$ and sides $= BD$ construct $\triangle ABC$.

Then, $\triangle ABC$ is the $\triangle$ required.

**Ex. 292.** Construct a rectangle, having given two adjacent sides.

**Solution.** Since the $\angle$s of a rectangle are rt. $\angle$s, this is a special case of § 248.

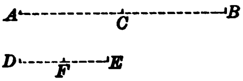
**Ex. 293.** Construct a rectangle, having given the shorter side and the difference of two sides.

**Solution.** Add the shorter side to the difference to obtain the longer side and proceed as in § 248.

**Ex. 294.** Construct a rectangle, having given the longer side and the difference of two sides.

**Solution.** Subtract the difference from the longer side to obtain the shorter side and proceed as in § 248.

**Ex. 295.** Construct a rectangle, having given the sum and difference of two adjacent sides.

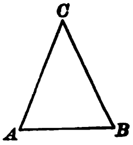


**Solution.** Draw $AB =$ sum of the given lines and find its middle pt. $C$; draw $DE =$ the difference of the given lines and find its middle pt. $F$; then, $AC$ and $DF$ are the longer and shorter sides respectively of the required rectangle. Proceed as in § 248.

**Ex. 296.** Construct a rhombus, having given one of its angles and the length of its side.

**Solution.** Since, § 144, the sides of a rhombus are equal, proceed as in § 248.

**Ex. 297.** Construct an isosceles triangle, having given the base and one of the two equal angles at the base.



**Solution.** Take $AB =$ the given base and construct $\angle$s $A$ and $B$ each $=$ the given $\angle$ producing the sides to intersect as at $C$.

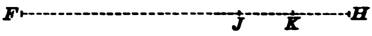
Then, $\triangle ABC$ is the $\triangle$ required.

**Ex. 298.** Construct an isosceles right triangle, having given its hypotenuse.

**Solution.** Construct $\triangle ABC$ as in Ex. 297, having the given $\angle = 45°$.

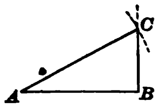
**Proof.** $\angle A = \angle B$, and $\angle A + \angle B = 90°$; $\therefore \angle C = 90°$; hence, $\triangle ABC$ is the $\triangle$ required.

**Ex. 299.** Construct a rhomboid, having given the perimeter, one side, and one angle.



**Solution.** Take $FH =$ given perimeter, and since the opposite sides are equal, lay off $FJ =$ double the given side, and find $K$ the middle pt. of $JH$; then, $KH =$ the other side of rhomboid. Proceed as in § 248.

**Ex. 300.** Construct a right triangle, having given the hypotenuse and one side.



**Solution.** Draw $AB =$ the given side and at $B$ erect a $\perp$.

From $A$ as a center with a radius equal to the given hypotenuse describe an arc cutting the $\perp$ as at $C$. Draw $AC$.

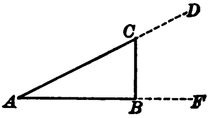
Then, $\triangle ABC$ is the $\triangle$ required.

**Ex. 301.** Construct a right triangle, having given the hypotenuse and one acute angle.

**Solution.** Construct $\angle DAF =$ the given $\angle$ and on $AD$ take $AC =$ the given hypotenuse.

From $C$ draw $CB \perp AF$.

Then, $\triangle ABC$ is the $\triangle$ required.



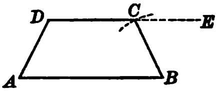
**Ex. 302.** Construct an isosceles trapezoid, having given two sides and the included angle.

**Solution.** Draw $AB =$ one of the given sides and construct $\angle A =$ given $\angle$, making $AD =$ the other given side.

Through $D$ draw $DE \parallel AB$.

From $B$ as center with radius $= AD$, describe an arc cutting $DE$ as at $C$. Draw $BC$.

Then, $ABCD$ is the trapezoid required.



**Proof.** Const., $AB \parallel DC$ and $AD = BC$; $\therefore$ $ABCD$ is an isosceles trapezoid and it has two sides $AB$ and $AD$ and the included $\angle =$ the given parts.

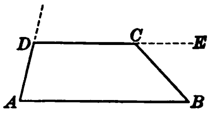
Hence, $ABCD$ is the trapezoid required.

**Ex. 303.** Construct a trapezoid, having given two adjacent sides, the included angle, and the angle at the other extremity of the given parallel base.

**Solution.** Draw $AB =$ one of the given sides for the base, construct $\angle A =$ the given included $\angle$, and take $AD =$ the other given side. Through $D$ draw $DE \parallel AB$.

Construct $\angle B =$ the other given $\angle$, and produce its side to intersect $DE$, as at $C$.

Then, $ABCD$ is the required trapezoid.



**Ex. 304.** From a given point without a given line, to draw a line making a given angle with the given line.

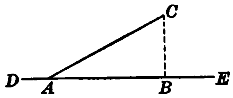
**Solution.** From the given pt. $C$ draw $CB \perp DE$ the given line, and, Ex. 282, construct $\angle C =$ the comp. of the given $\angle$, producing the side to meet $DE$ as at $A$.

Then, $CA$ is the required line.

**Proof.** § 111, $\angle CAB$ is comp. of $\angle C$, and, const., the given $\angle$ is comp. of $\angle C$;

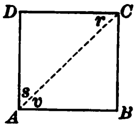
$\therefore$ § 54, $\angle CAB =$ given $\angle$.

Hence, $CA$ makes the given $\angle$ with $DE$.



**Ex. 305.** Construct a square, having given a diagonal.

**Solution.** Draw $AC =$ the given diagonal; on opposite sides of $AC$, Ex. 298, construct isos. rt. $\triangle$s $ABC$ and $ADC$.



Then, $ABCD$ is the required square.

**Proof.** Since $\angle r = 45°$, and $\angle v = 45°$, § 75, $DC \parallel AB$; similarly, $AD \parallel BC$; hence, § 140, $ABCD$ is a ▱, § 153, $AB = DC$, and $BC = AD$; $\therefore$ since $AD = DC$, all the sides of $ABCD$ are equal.

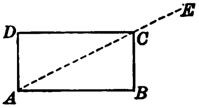
Const., $\angle B$ is a rt. $\angle$; $\therefore$ Ex. 104, all the $\angle$s of $ABCD$ are rt. $\angle$s.

Hence, § 143, $ABCD$ is a square.

Ex. **306**. Construct a rectangle, having given one side and the angle included between it and a diagonal.

**Solution.** Draw $AB =$ the given side, and at $A$ construct $\angle BAE =$ the given $\angle$.

At $B$ draw $BC \perp AB$, meeting $AE$ at $C$.



From $C$ and $A$ as centers with radii equal respectively to $AB$ and $BC$, describe arcs intersecting as at $D$.

Draw $AD$ and $DC$.

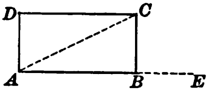
Then, $ABCD$ is the rectangle required.

**Proof.** Const., the opposite sides of $ABCD$ are equal; $\therefore$ § 148, $ABCD$ is a parallelogram, and since, const., $\angle B$ is a rt. $\angle$, Ex. 104, the other $\angle$s of $ABCD$ are rt. $\angle$s; $\therefore$ §141, $ABCD$ is a rectangle.

Ex. **307**. Construct a rectangle, having given a diagonal and an angle between it and a side.

**Solution.** Draw $AC =$ the given diag., and construct $\angle CAE =$ the given $\angle$.

Draw $CB \perp AE$.



From $A$ and $C$ as centers with radii respectively equal to $BC$ and $AB$, describe arcs intersecting as at $D$, and draw $AD$ and $CD$.

Then, $ABCD$ is the rectangle required.

**Proof.** The same as for Ex. 306.

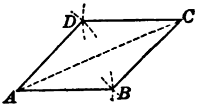
Ex. **308**. Construct a rhombus, having given its perimeter and one diagonal.

**Solution.** Draw $AC =$ the given diagonal.

From $A$ and $C$ as centers with radii each equal to ¼ the given perimeter describe arcs intersecting as at $D$ and $B$.

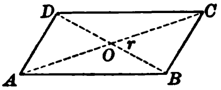
Draw $AB$, $BC$, $CD$, and $AD$.

Then, $ABCD$ is the required rhombus.



**Proof.** Const., the opposite sides of $ABCD$ are equal; $\therefore$ § 148, $ABCD$ is a ▱, and since all of its sides are equal, § 144, $ABCD$ is a rhombus.

**Ex. 309.** Construct a rhomboid, having given two diagonals and an angle between them.

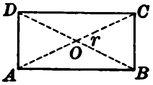


**Solution.** Draw $AC =$ one of the given diagonals.

Since, § 154, the diagonals must bisect each other, at $O$, the middle point of $AC$, construct $\angle r$ = the given $\angle$, making side $OB = \frac{1}{2}$ the other given diagonal, and produce $BO$ to $D$, making $OD = BO$. Draw $AB$, $BC$, $CD$, and $AD$.

Then, $ABCD$ is the required rhomboid.

**Ex. 310.** Construct a rectangle, having given one diagonal and the angle included between the two diagonals.



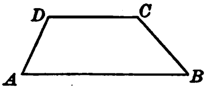
**Solution.** Since, Ex. 112, the diagonals of a rectangle are equal, construct $ABCD$ as in Ex. 309, making each diagonal = the given diagonal.

Then, Ex. 100, $ABCD$ is the required rectangle.

**Ex. 311.** Construct a rectangle, having given the perimeter and one side.

**Solution.** Find the other side as in Ex. 299, and proceed as in § 248.

**Ex. 312.** Construct a trapezoid, having given two sides, the included angle, and the difference between the two parallel sides.

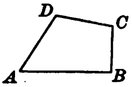


**Solution.** Draw $AB =$ one of the given sides; construct $\angle A =$ the given $\angle$, making side $AD =$ the other given side. Draw $DC \parallel AB$ and equal to $AB$ less the difference of the parallel sides. Draw $BC$.

Then, $ABCD$ is the required trapezoid.

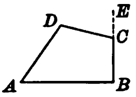
**Sch.** If $DC$ is taken equal to $AB$ plus the given difference, there will be another solution.

**Ex. 313.** Construct a trapezium, having given three consecutive sides and the two included angles.



**Solution.** Draw $AB =$ the middle given side, and construct $\angle$s $A$ and $B =$ respectively to the given $\angle$s, making the sides $AD$ and $BC =$ respectively to the other given sides taken in the order corresponding to that in which the $\angle$s adjacent to them were selected. Draw $DC$. Then, $ABCD$ is the required trapezium.

**Ex. 314.** Construct a trapezium, having given two adjacent sides and the three angles adjacent to these sides.



**Solution.** Draw $AB =$ one of the given sides; construct $\angle A =$ the $\angle$ included by the given sides, making $AD =$ the other given side; construct $\angle$s $B$ and $D =$ respectively to the other given $\angle$s taken in the order corresponding to that in which the sides adjacent to them were selected, the sides $BE$ and $DC$ meeting, as at $C$.

Then, $ABCD$ is the required trapezium.

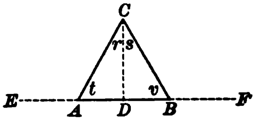
**Ex. 315.** **Construct an isosceles triangle, having given the base and the vertical angle.**

**Solution.** Construct the supplement of the given $\angle$ and bisect it.

Then, with a base $\angle =$ one of these parts, construct isos. $\triangle ABC$, as in Ex. 297.

Then, $\triangle ABC$ is the $\triangle$ required.

**Ex. 316.** **Construct an equilateral triangle, having given its altitude.**



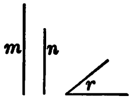
**Solution.** Draw an indefinite line $EF$, and at some point in it, as $D$, draw $CD \perp EF$ and equal to the given altitude. Ex. 287, construct $\angle$s $r$ and $s$ each $= \frac{1}{3}$ rt. $\angle$, and produce the sides $CA$ and $CB$ to meet $EF$ as in $A$ and $B$, respectively.

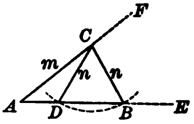
Then, $\triangle ABC$ is the $\triangle$ required.

**Proof.** In rt. $\triangle ADC$, § 111, $\angle t = 1$ rt. $\angle - \angle r = \frac{2}{3}$ rt. $\angle$; similarly, $\angle v = \frac{2}{3}$ rt. $\angle$, and $\angle ACB = \angle r + \angle s = \frac{2}{3}$ rt. $\angle$;

$\therefore$ $\triangle ABC$ is equiangular; hence, § 119, $\triangle ABC$ is equilateral.

**Ex. 317.** **Construct a triangle, having given two sides and the angle opposite one.**

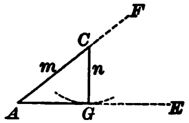




**Solution.** Let $m$ and $n$ be the given sides and $r$ the given $\angle$.

**Case I.** *When m is greater than n.*

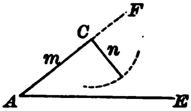
Construct $\angle FAE = \angle r$; on $AF$ take $AC = m$; from $C$ as center with radius $= n$ describe an arc intersecting $AE$ in $B$ and $D$; and draw $BC$ and $DC$.



Then, either $\triangle ABC$ or $\triangle ADC$ fulfills the conditions, and there are two solutions.

This is called the *ambiguous case.*

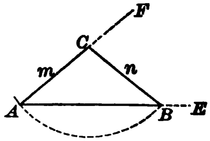
**Sch.** If the side $n$ is equal to the perpendicular $CG$, the arc whose center is $C$ will touch $AE$ and there will be only one solution, the rt. $\triangle AGC$.



If the side $n$ is less than the $\perp$ from $C$ to $AE$, the arc whose center is $C$ will neither intersect nor touch $AE$, and consequently, the problem is impossible.

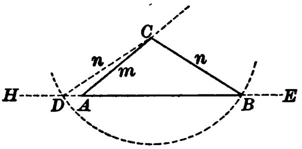
If $\angle r$ is right or obtuse, since the side opposite is greatest, the problem is impossible.

**Case II.** *When m is equal to n.*



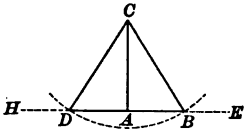
If $\angle r$ is acute, the arc whose center is $C$ will intersect $AE$ at $A$ and some other point as $B$, and there is only one solution, the isos. $\triangle ABC$.

**Sch.** Since equal sides of a $\triangle$ have equal $\angle$s opposite them and since a $\triangle$ can have only one rt. $\angle$ or one obtuse angle, if $\angle r$ is rt. or obtuse the problem is impossible.

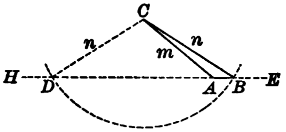


**Case III.** *When m is less than n.*

If $\angle r$ is acute, the arc whose center is $C$ will intersect $HE$ on opposite sides of $A$, as at $B$ and $D$, but only $\triangle ABC$ fulfills the conditions, for the $\triangle ADC$ does not contain the acute $\angle r$; consequently, there is only one solution.

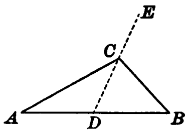


If $\angle r$ is a rt. $\angle$, the arc whose center is $C$ will intersect $HE$ on opposite sides of $A$, as at $B$ and $D$, and there are two solutions, the *equal* rt. $\triangle$s $ABC$ and $ADC$, fulfilling the required conditions.



If $\angle r$ is obtuse, the arc whose center is $C$ will intersect $HE$ on opposite sides of $A$, as at $B$ and $D$, but only $\triangle ABC$ fulfills the conditions, for the $\triangle ADC$ does not contain the obtuse $\angle r$; consequently, there is only one solution.

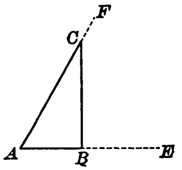
Ex. **318.** Construct a triangle, having given its base, the median to the base, and the angle included between them.



**Solution.** Draw $AB =$ the given base, and at $D$ its middle point construct $\angle BDE =$ the given $\angle$.

On $DE$ lay off $DC =$ the given median, and draw $AC$ and $BC$. Then, $\triangle ABC$ is the $\triangle$ required.

Ex. **319.** Construct a right triangle, having given its hypotenuse and having one of its acute angles double the other.

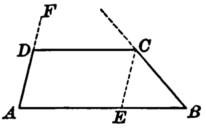


**Solution.** Construct $\angle EAF = 60°$; on $AF$ take $AC$ = the given hypotenuse; and from $C$ draw $CB \perp AE$.

Then, $\triangle ABC$ is the $\triangle$ required.

**Proof.** Const., $ABC$ is a rt. $\triangle$ whose hypotenuse $AC =$ the given hypotenuse, and since $\angle A = 60°$, § 111, $\angle C = 90° - 60° = 30°$; hence, $\angle A = 2\angle C$.

Ex. **320.** Construct a trapezoid, having given the sum and difference of the parallel sides, and the sum and difference of the angles at the base.



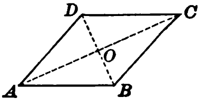
**Solution.** Find the parallel sides and the angles at the base by the method exemplified in Ex. 295.

Draw $AB =$ the longer parallel side, and construct $\angle$s $BAF$ and $B =$ respectively the base $\angle$s.

On $AB$ take $AE =$ the shorter parallel side and draw $EC \parallel AF$ meeting side of $\angle B$ as at $C$. Draw $CD \parallel BA$, meeting $AF$ as at $D$.

Then, $ABCD$ is the required trapezoid.

Ex. **321**. Construct a rhombus, having given its diagonals.



**Solution.** Draw $AC =$ a given diagonal, and draw $BD =$ the other diagonal, so that $AC$ and $BD$ bisect each other at rt. $\angle$s. Draw $AB$, $BC$, $AD$, and $DC$.

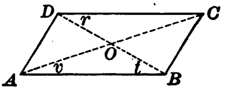
Then, $ABCD$ is the required rhombus.

**Proof.** Since, Ex. 111, the diagonals of a rhombus bisect each other at rt. $\angle$s, const., $AC$ and $BD$ are the diagonals given in position, and their extremities are the angular points of the required rhombus.

Ex. **322**. Construct a rhomboid, having given two adjacent sides and an angle not included by them.

**Solution.** Construct the supplement of the given $\angle$, and with this as the included $\angle$ proceed as in § 248.

Ex. **323**. Construct a rhomboid, having given one side and the angles included between it and the diagonals.



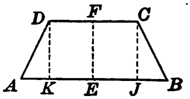
**Solution.** Draw $AB =$ given side, and construct $\angle$s $v$ and $t =$ respectively the given $\angle$s, producing the sides to intersect as at $O$. Produce $BO$ to $D$, making $OD = BO$, and $AO$ to $C$ making $OC = AO$.

Draw $AD$, $DC$, and $BC$.

Then, $ABCD$ is the required rhomboid.

**Proof.** Const. and § 100, $\triangle ABO = \triangle DOC$, and $AB = DC$; and since, § 108, $\angle t = \angle r$, § 75, $AB \parallel DC$; $\therefore$ § 150, $ABCD$ is a ▱, fulfilling the given conditions, and is therefore the required rhomboid.

Ex. **324**. Construct an isosceles trapezoid, having given the bases and the altitude.



**Solution.** Draw $AB =$ the longer given base; at its middle point draw $EF \perp AB$ and $=$ the given altitude; and through $F$ draw $DC \parallel AB$, making $DF = FC = \frac{1}{2}$ the shorter given base. Draw $AD$ and $BC$. Then, $ABCD$ is the trapezoid required.

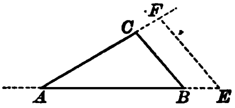
**Proof.** Const., $DC \parallel AB$; $\therefore$ § 138, $ABCD$ is a trapezoid.

Draw $DK$ and $CJ \perp AB$; then, § 71, $DK \parallel FE \parallel CJ$, and, § 157, $KE = EJ$, and since, const., $AE = EB$, Ax. 3, $AK = JB$, also, § 151, $DK = CJ$, and, § 52, $\angle AKD = \angle CJB$; $\therefore \triangle AKD = \triangle BJC$, and $AD = BC$; hence, § 139, $ABCD$ is an isosceles trapezoid.

Ex. **325**. Construct an isosceles trapezoid, having given the altitude and the sum and difference of the parallel sides.

**Solution.** From the given lines find the parallel sides, as in Ex. 295; then proceed as in Ex. 324.

**Ex. 326.** Construct a triangle, having given two angles and a side opposite one.



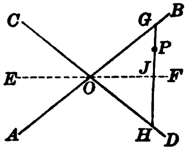
**Solution.** At any points, as $A$ and $E$, in an indefinite line construct $\angle$s $EAF$ and $AEF$ = respectively to the given $\angle$s, their exterior sides meeting as at $F$, and lay off $AC$ on $AF$ (or $AF$ produced) = the given side.

Through $C$ draw $CB \parallel FE$, and meeting the indefinite line as in $B$.

Then, $\triangle ABC$ is the required $\triangle$.

**Proof.** § 76, $\angle ABC = \angle AEF =$ given $\angle$ which the given side is opposite.

**Ex. 327.** Draw a line which shall pass through a given point and make equal angles with two given intersecting lines.



**Solution.** Draw $EF$ to bisect a pair of vert. $\angle$s formed by the given lines $AB$ and $CD$ intersecting at $O$.

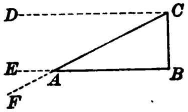
Through the given point $P$ draw $PJ \perp EF$, and produce it to intersect $AB$ and $CD$ in $G$ and $H$ respectively.

Then, $GH$ is the required line.

**Proof.** In rt. $\triangle$s $OJG$ and $OJH$, const., $\angle GOJ = \angle HOJ$;

$\therefore$ § 113, $\angle OGJ = \angle OHJ$.

**Ex. 328.** Construct a right triangle, having given one side and the angle opposite.



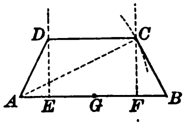
**Solution.** Draw $BC$ = the given side; draw $CD$ and $BE \perp BC$ on the same side of it; construct $\angle DCF$ = the given $\angle$, the side $CF$ intersecting $BE$ as in $A$.

Then, $\triangle ABC$ is the required $\triangle$.

**Proof.** § 73, const., $\angle BAC = \angle ACD =$ the given $\angle$.

**Ex. 329.** Construct an isosceles trapezoid, having given the bases and a diagonal.

**Solution.** Draw $AB$ = the longer base, find its middle point $G$, and lay off $GE = GF = \frac{1}{2}$ the shorter base; at $E$ and $F$ erect $\perp$s to $AB$; from $A$ as a center with a radius = the given diagonal, describe an arc to cut the $\perp$ from $F$, as at $C$; draw $CD \parallel BA$, meeting the other $\perp$, as at $D$; draw $AD$ and $BC$.



Then, $ABCD$ is the trapezoid required.

**Proof.** Const., $DC \parallel AB$; $\therefore$ $ABCD$ is a trapezoid.

§§ 71, 151, $DE = CF$,

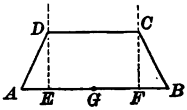
Ax. 3, $AG - EG = BG - FG$, or $AE = BF$,

and, § 52, $\angle AED = \angle BFC$;

$\therefore$ § 100, $\triangle AED = \triangle BFC$, and $AD = BC$.

Hence, § 139, $ABCD$ is an isosceles trapezoid.

**Ex. 330.** Construct an isosceles trapezoid, having given the bases and one angle.



**Solution.** Draw $AB =$ the longer base, find its middle point $G$, and lay off $GE = GF = \frac{1}{2}$ the shorter base; at $E$ and $F$ erect ⊥s to $AB$; construct ∠s $A$ and $B =$ the given ∠, and produce their exterior sides to intersect the ⊥s erected at $E$ and $F$ in $D$ and $C$, respectively. Draw $DC$.

Then, $ABCD$ is the required trapezoid.

**Proof.** Ax. 3, $AG - EG = BG - FG$, or $AE = BF$,

and, const., $\angle A = \angle B$;

∴ § 114, rt. $\triangle AED =$ rt. $\triangle BFC$, and $ED = FC$;

also, § 71, $ED \parallel FC$;

hence, § 150, $EFCD$ is a ▱,

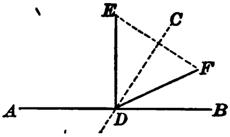
and, § 153, $DC = EF =$ the given shorter base.

Also, § 140, $DC \parallel EF$, or $AB$,

and, § 108, $AD = BC$.

Hence, $ABCD$ is an isosceles trapezoid.

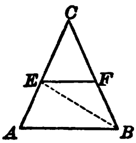
**Ex. 331.** From two given points, draw two equal straight lines which shall meet in the same point of a line given in position.



**Solution.** Join the given points $E$ and $F$ by the line $EF$, and draw $CD$, the perpendicular bisector of $EF$, meeting the given line $AB$ as in $D$. Draw $ED$ and $FD$.

Then, § 103, $ED$ and $FD$ are the required lines.

**Ex. 332.** $ABC$ is an isosceles triangle. Draw a straight line parallel to the base $AB$, and meeting the equal sides in $E$ and $F$, so that $BF$, $EF$, and $EA$ are all equal.



**Solution.** Bisect $\angle B$ and produce the bisector to meet $AC$, as at $E$. Draw $EF \parallel AB$, meeting $BC$ as in $F$.

Then, $BF = EF = EA$.

**Proof.** Const., $\angle EBF = \angle EBA$,

and, § 73, $\angle BEF = \angle EBA$;

∴ $\angle BEF = \angle EBF$,

and, § 118, $BF = EF$.

§ 76, $\angle CEF = \angle A$, and $\angle CFE = \angle CBA$;

but $\angle A = \angle CBA$; ∴ $\angle CEF = \angle CFE$,

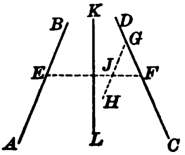
and, § 118, $EC = FC$;

but $AC = BC$;

∴ Ax. 3, $EA = BF$;

hence, $BF = EF = EA$.

**Ex. 333.** Given two straight lines which cannot be produced to their intersection, to draw a third line which would pass through their intersection and bisect their contained angle.



**Solution.** Through any point $G$ of one given line $CD$, draw $HG \parallel AB$, the other given line; on $GC$ and $GH$ lay off $GF$ and $GJ$, respectively, making $GF = GJ$.

Draw $FJ$, and produce it to meet $AB$, as at $E$.

Draw $LK$, the perpendicular bisector of $EF$.

Then, $LK$ is the required line.

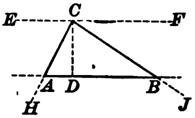
**Proof.** § 116, $\angle JFG = \angle FJG$,

§ 76, $\angle BEF = \angle FJG$;

$\therefore$ $\angle BEF = \angle JFG$;

hence, § 118, if $AB$ and $CD$ were produced to their intersection, they would form with $EF$ an isosceles $\triangle$, and, § 121, $LK$ would pass through their intersection and bisect their contained angle.

**Ex. 334.** Construct a triangle, having given the altitude and the angles at the base.



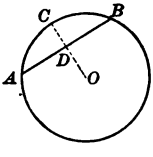
**Solution.** Draw $CD =$ the given altitude; through $C$ draw $EF \perp CD$; and on the same side of $EF$ as $D$ construct $\angle$s $ECH$ and $FCJ =$ respectively the given $\angle$s.

Through $D$ draw a line $\perp$ $CD$ and intersecting $CH$ and $CJ$ in $A$ and $B$, respectively.

Then, $\triangle ABC$ is the $\triangle$ required.

**Proof.** § 71, $AB \parallel EF$; $\therefore$ § 73, $\angle$s $CAB$ and $ABC =$ respectively $\angle$s $ECH$ and $FCJ =$ respectively the given $\angle$s.

**Ex. 335.** Given the middle point of a chord in a given circle, to draw the chord.

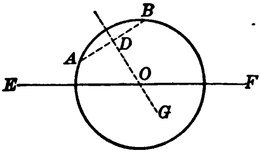


**Solution.** Draw the radius $OC$ through the given point $D$; draw a line perpendicular to $OC$ at $D$ meeting the circumference in $A$ and $B$.

Then, $AB$ is the required chord.

**Proof.** Since, const., $OC \perp AB$ at $D$, § 198, $D$ is the middle point of $AB$.

**Ex. 336.** Construct a circle to pass through two given points and have its center on a given straight line. When is this impossible?



**Solution.** Join the given points $A$ and $B$ by the straight line $AB$; draw the perpendicular bisector of $AB$, as $GD$, and produce it, if necessary, to intersect the given line $EF$ in a point $O$.

With $O$ as center and radius $= OA$, describe a $\odot$.

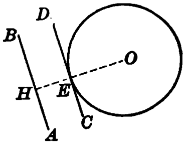
Then, this $\odot$ is the $\odot$ required.

**Proof.** $O$ is in $EF$, and, § 103, $OA = OB$; $\therefore$ the constructed $\odot$ has its center in $EF$ and passes through $A$ and $B$.

**Sch.** If $GD$ and $EF$ coincide, any point in $EF$ may be taken as center and there may be an infinite number of solutions.

If $GD$ is parallel to $EF$, the problem is impossible.

Ex. **337.** Draw a tangent to a circle parallel to a given straight line.



**Solution.** From $O$, the center of the $\odot$, draw $OH \perp AB$, the given line, and cutting the circumference as at $E$.

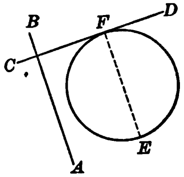
Through $E$ draw $CD \perp OH$.

Then, $CD$ is the required tangent.

**Proof.** § 71, $CD \parallel AB$, and, § 205, $CD$ is tangent to the $\odot$.

**Sch.** Another solution may be obtained by drawing a line perpendicular to the diameter from $E$, at its other extremity.

Ex. **338.** Draw a tangent to a circle perpendicular to a given straight line.

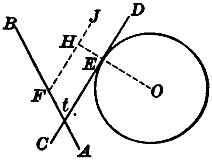


**Solution.** Draw the diameter $EF \parallel AB$, the given line, and through $F$ draw $CD \perp EF$.

Then, $CD$ is the required tangent.

**Sch.** A line $\perp EF$ at $E$ would also fulfill the conditions.

Ex. **339.** Draw a straight line tangent to a given circle and making with a given line a given angle.



**Solution.** Draw $FJ$ to make with the given line $AB$ the $\angle JFB =$ the given $\angle$. By Ex. 337, draw $CD$ tangent to the $\odot$ and parallel to $FJ$.

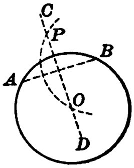
Then, $CD$ is the required tangent.

**Proof.** Denote the $\angle$ between $CD$ and $AB$ by $t$.

Then, § 76, $\angle t = \angle JFB =$ the given $\angle$.

**Sch.** As in Ex. 337, there may be two solutions.

Ex. **340.** Construct a circle of given radius to pass through two given points. When is this impossible?



**Solution.** Join the given points $A$ and $B$ by the straight line $AB$ and draw its perpendicular bisector $CD$.

With the given radius and $B$ as a center describe an arc cutting $CD$ as at $O$ and $P$.

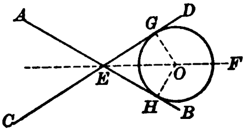
With the given radius and $O$ as a center describe a $\odot$.

Then, § 103, the $\odot$ whose center is $O$ is the required $\odot$.

**Sch.** With $P$ as center and the given radius a second $\odot$ may be described fulfilling the conditions.

If the given radius is less than half $AB$ the problem is impossible.

**Ex. 341.** Construct a circle tangent to two intersecting lines with its center at a given distance from their intersection. How many such circles can be drawn?



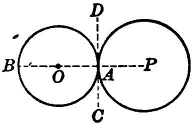
**Solution.** Bisect the ∠ between the given lines, $AB$ and $CD$, intersecting in $E$, by the line $EF$; on $EF$ lay off $EO =$ the given distance; draw $OH \perp AB$, and $OG \perp CD$.

With $O$ as center and radius equal $OG$, describe a ⊙.

Then, § 205, this ⊙ is the ⊙ required.

**Sch.** There may be four such Ⓢ, the opposite ones being equal.

**Ex. 342.** From a given point as a center, to describe a circle tangent to a given circle. How many solutions may there be?



**Solution.** Draw a line from $O$, the center of the given ⊙, to $P$ the given point, cutting the circumference at $A$.

With $P$ as center and radius $PA$, describe a ⊙.

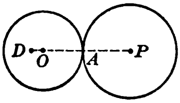
Then, this ⊙ is the ⊙ required.

**Proof.** Draw $CD \perp OP$ at point $A$; then, § 205, $CD$ is tangent to each of the Ⓢ, and, § 185, the ⊙ whose center is $P$ is tangent to the given ⊙.

**Sch.** If $PO$ is produced to meet the circumference of the given ⊙ as at $B$, a ⊙ with $P$ as center and $PB$ as radius will also fulfill the conditions.

If $P$ is on the circumference of the given ⊙, there is only one solution.

**Ex. 343.** Construct a circle of given radius tangent to a given circle at a given point. How many solutions may there be?



**Solution.** Draw $OA$ the radius of the given ⊙ to the given point $A$ and produce it to $P$ making $AP =$ the given radius.

Describe the ⊙ whose center is $P$ and radius $PA$.

Then, this ⊙ is the ⊙ required.

**Proof.** The same as for Ex. 342.

**Sch.** If $AO$ is produced to $D$, making $AD =$ the given radius, the ⊙ described from $D$ as center with $DA$ as radius will also fulfill the conditions.

If the given radius equals the radius of the given ⊙, one of the resulting Ⓢ will coincide with the given ⊙ and there will be only one solution.

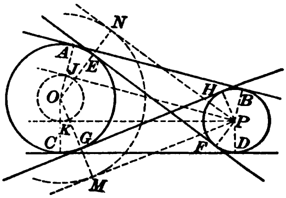
**Ex. 344.** Draw a common tangent to two given circles. How many solutions may there be?

**Solution.** Let $O$ and $P$ be the centers of the given Ⓢ whose radii are $r$ and $r'$ respectively.

With $O$ as center and radius $r - r'$, describe a ⊙; from $P$ draw to this ⊙ the tangents $PJ$ and $PK$.

Draw $OJ$ and $OK$ and produce them to meet the given circumference as at $A$ and $C$ respectively; draw the radii $PB$ and $PD \parallel OA$ and $OC$ respectively; and draw straight lines through $A$ and $B$ and through $C$ and $D$.

Then, $AB$ and $CD$ are common tangents to the given Ⓢ.



**Proof.** Const., $JA = PB$, and $JA \parallel PB$; $\therefore$ § 150, $ABPJ$ is a ▱, and $AB \parallel JP$, and since, § 205, $JP \perp OA$, § 72, $AB \perp OA$, also $AB \perp PB$; hence, § 205, $AB$ is tangent to the given Ⓢ. Similarly, $CD$ is tangent to the given Ⓢ.

**Sch.** If the distance $OP > r + r'$, two other common tangents $EF$ and $GH$ may be drawn by using $r + r'$ instead of $r - r'$ for the radius of the auxiliary ⊙ whose center is $O$.

$AB$ and $CD$ are the *exterior tangents*, and $EF$ and $GH$ the *interior tangents*.

If $OP = r + r'$, the Ⓢ are tangent to each other externally, and there are only three solutions.

If $OP < r + r'$ and $> r - r'$, the Ⓢ intersect, and there are only two solutions.

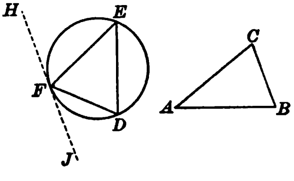
If $OP = r - r'$, the Ⓢ are tangent to each other internally, and there is only one solution.

If $OP < r - r'$, one ⊙ is entirely within the other, and the problem is impossible.

Ex. **345.** In a given circle, to inscribe a triangle equiangular to a given triangle.

**Solution.** Draw $HJ$ tangent to the given ⊙ at $F$; draw chord $FD$ making $\angle DFJ = \angle A$ of the given $\triangle ABC$, and chord $FE$ making $\angle EFH = \angle B$. Draw $DE$.

Then, $\triangle DEF$ is the $\triangle$ required.



**Proof.** §§ 225, 231,

$$\angle FED = \angle DFJ = \angle A,$$

and $$\angle FDE = \angle EFH = \angle B;$$

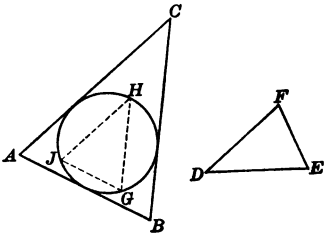
$\therefore$ § 113, $$\angle DFE = \angle C.$$

Ex. **346.** About a given circle, to circumscribe a triangle equiangular to a given triangle.

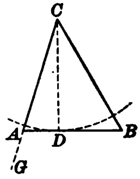
**Solution.** By Ex. 345, construct inscribed $\triangle GHJ$ equiangular to the given $\triangle DEF$.

By Ex. 337, draw tangents $AB$, $BC$, and $AC$ parallel respectively to $JG$, $GH$, and $JH$, and intersecting in points $A$, $B$, and $C$.

Then, $\triangle ABC$ is the $\triangle$ required.



Ex. **347.** Construct a triangle, having given the vertical angle, one of the sides containing it, and the altitude.



**Solution.** Construct $\angle GCB =$ the given $\angle$, making side $CB =$ the given side.

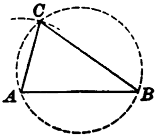
With $C$ as center and radius = the given altitude, describe an arc cutting the sides of $\angle GCB$.

Through $B$ draw a tangent to this arc as at $D$, meeting $GC$ as in $A$.

Then, $\triangle ABC$ is the $\triangle$ required.

**Proof.** Draw $CD$; then, § 205, $CD \perp AB$, and equal the given altitude.

Ex. **348.** Construct a triangle, having given the base, the vertical angle, and one other side.



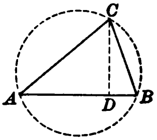
**Solution.** § 253, on $AB =$ the given base describe a segment that will contain the given $\angle$.

With $A$ as center, and radius = the given side, describe an arc cutting the arc of the segment, as at $C$.

Draw $AC$ and $BC$.

Then, $\triangle ABC$ is the $\triangle$ required.

Ex. **349.** Construct a triangle, having given the base, the vertical angle, and the foot of the perpendicular from the vertex to the base.

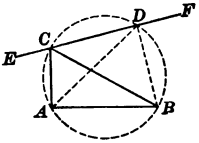


**Solution.** On $AB =$ the given base describe a segment that will contain the given $\angle$.

At $D$, the given point in $AB$, erect a $\perp$ to $AB$, intersecting the arc of the segment, as at $C$. Draw $AC$ and $BC$.

Then, $\triangle ABC$ is the $\triangle$ required.

Ex. **350.** Construct a triangle whose vertex is on a given straight line, and having given its base and vertical angle.



**Solution.** On the given base $AB$ describe a segment that will contain the given $\angle$, the arc intersecting the given line $EF$ in $C$ and $D$.

Draw $AC$ and $BC$.

Then, $\triangle ABC$ is the $\triangle$ required.

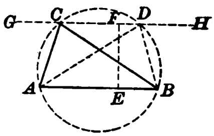
**Sch.** $\triangle ABD$ also fulfills the required conditions.

If $EF$ is tangent to the arc there is only one solution.

If $EF$ is beyond the arc there is no solution.

Ex. **351.** Construct a triangle, having given the base, the vertical angle, and the altitude.

**Solution.** On $AB =$ the given base describe a segment that will contain the given $\angle$.



At any point in $AB$, as $E$, erect the $\perp EF =$ the given altitude.

Through $F$ draw $GH \parallel AB$ and intersecting the arc of the segment in $C$ and $D$. Draw $AC$ and $BC$.

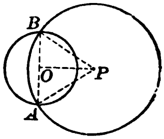
Then, $\triangle ABC$ is the $\triangle$ required.

**Sch.** $\triangle ABD$ also fulfills the required conditions.

If the altitude of the segment equals the given altitude, $GH$ is tangent to the arc, and there is only one solution.

If the altitude of the segment is less than the given altitude, there is no solution.

**Ex. 352.** Describe a circle with a given center to intersect a given circle at the extremities of a diameter. Is this ever impossible?



**Solution.** Draw a line from $O$, the center of the given $\odot$, to $P$, the given center; draw $AB$, the diameter of the given $\odot$, perpendicular to $OP$.

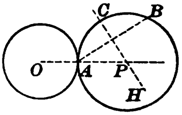
With $P$ as center and radius $PA$ describe a $\odot$.

Then, this $\odot$ is the $\odot$ required.

**Proof.** Since $OA = OB$, § 103, $PA = PB$, and the $\odot$ whose center is $P$ and radius $PA$ passes through $B$ and is the $\odot$ required.

**Sch.** If $P$ is within the given $\odot$, the problem is impossible.

**Ex. 353.** Construct a circle to pass through a given point and be tangent to a given circle at a given point. When is this impossible?



**Solution.** Draw the radius from $O$, the center of the given $\odot$, to $A$, the given point of tangency; draw a line $AB$ joining $A$ and the other given point; and draw $CH$, the perpendicular bisector of $AB$, cutting $OA$ produced, as at $P$.

With $P$ as center and radius $PA$ describe a $\odot$.

Then, this $\odot$ is the $\odot$ required.

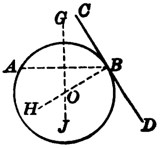
**Proof.** Since, § 103, $PA = PB$, the circle whose center is $P$ and radius $PA$ passes through $B$.

Also, it may be shown that the constructed $\odot$ is tangent to the given $\odot$ at $A$, as in Ex. 342; consequently, it is the required $\odot$.

**Sch.** If $B$ is on the circumference of the given $\odot$, $P$ will coincide with $O$ and the $\odot$ described will coincide with the given $\odot$.

If $AB$ is perpendicular to $OP$, the problem is impossible.

**Ex. 354.** Construct a circle to pass through a given point and touch a given straight line at a given point.



**Solution.** At $B$ the given point in the given line $CD$ draw $BH \perp CD$; from $A$, the other given point, draw $AB$; and draw $GJ$, the perpendicular bisector of $AB$, intersecting $BH$ as in $O$.

With $O$ as center and radius $OB$ describe a ⊙.

Then, this ⊙ is the ⊙ required.

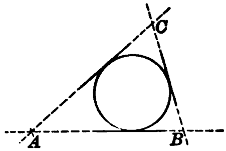
**Proof.** Since, § 103, $OA = OB$, the ⊙ whose center is $O$ and radius $OB$ passes through $A$, and, § 205, touches $CD$ at $B$.

**Sch.** When $A$ is in $CD$ the problem is impossible.

Ex. **355.** Construct a circle to touch three given straight lines.

**Solution.** If the lines do not intersect produce them to intersect as in $A$, $B$, and $C$ forming the $\triangle ABC$.

Then, § 249, draw the inscribed ⊙ of $\triangle ABC$, which is the ⊙ required.



**Sch.** If the lines are produced through their intersections and the external ∡ bisected, the intersections of the bisectors will be the centers of three other Ⓢ that fulfill the conditions.

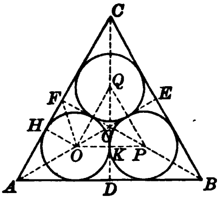
If two of the lines are parallel, there are only two Ⓢ that fulfill the conditions, their centers being the intersections of the bisectors of the two pairs of interior ∡ formed with the third line.

If the three lines are parallel, the problem is impossible.

Ex. **356.** Within an equilateral triangle, to describe three circles each tangent to the other two and to two sides of the triangle.

**Solution.** Since each ⊙ is to be tangent to two sides of the △ its center must lie in the bisector of the ∠ formed by those sides; consequently, draw the bisectors $AE$, $BF$, and $CD$ of the ∡ $A$, $B$, and $C$ of the $\triangle ABC$, and let them intersect as at $G$.

Draw $FO$ to bisect $\angle AFG$ and meet $AG$, as at $O$; draw $OH \perp AF$, and with $O$ as center and radius $OH$ describe a circle. Similarly, find the centers $P$ and $Q$ of Ⓢ described in $DBEG$ and $CFGE$.



Then, the Ⓢ whose centers are $O$, $P$, and $Q$ are the Ⓢ required.

**Proof.** Since $O$ is equidistant from $AD$ and $AF$ and also from $FG$ and $AF$ it is equidistant from $AD$, $AF$, and $FG$.

§ 120, the bisectors of the ∡ of $\triangle ABC$ are the ⊥s and medians of $\triangle ABC$.

Then, in rt. ▲ $ADG$ and $AFG$, const., $\angle DAG = \angle FAG$, and $AG$ is common; $\therefore \triangle ADG = \triangle AFG$, and $\angle AGD = \angle AGF$; that is, $AG$ bisects $\angle FGD$ and $O$ is equidistant from $FG$ and $DG$; consequently, the ⊙ whose center is $O$ is inscribed in $ADGF$.

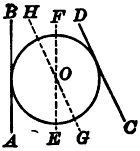
Similarly, the Ⓢ whose centers are $P$ and $Q$ are inscribed in $DBEG$ and $CFGE$ respectively.

On $DG$ as an axis fold over the figure $ADGF$ and apply it to the figure $DBEG$.

Since $\angle ADG = \angle GDB$, $DA$ falls on $DB$, and since these lines are equal they coincide; similarly $AF$ coincides with $BE$, and consequently, Ax. 11, $FG$ coincides with $EG$, and the figures coincide in all of their parts; then their inscribed Ⓢ must coincide; also the points of tangency of these Ⓢ with the sides coincide, consequently, when the figure is folded back to its original position the Ⓢ whose centers are $O$ and $P$ are tangent to $DG$ at the same point $K$; hence, § 185, the Ⓢ are tangent to each other.

Similarly, it may be shown that any two of the Ⓢ are tangent to each other; consequently, the Ⓢ whose centers are $O$, $P$, and $Q$ are tangent to each other and to two sides of the $\triangle ABC$.

**Ex. 357.** Construct a circle of given radius to touch two given straight lines.



**Solution.** Draw $EF \parallel AB$, one of the given lines, and at a distance from $AB$ equal to the given radius.

Draw $GH \parallel CD$, the other given line, and at a distance from $CD$ equal to the given radius.

Let $EF$ and $GH$ intersect, as at $O$.

With the given radius and $O$ as a center describe a ⊙.

Then, this ⊙ is the ⊙ required.

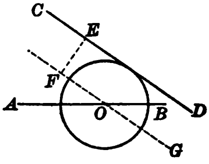
**Proof.** Const., § 151, every point in $EF$ is at the distance of the given radius from $AB$, and every point in $GH$ is at the distance of the given radius from $CD$; then, $O$ the intersection of $EF$ and $GH$ is equidistant from $AB$ and $CD$; hence, the ⊙ whose center is $O$ and radius the given radius will touch $AB$ and $CD$.

**Sch.** Since there may be parallels on each side of each of the given lines, there may be four possible intersections, and, consequently, four possible solutions.

If $AB$ and $CD$ are parallel there is no solution unless the given radius equals one half the perpendicular distance between $AB$ and $CD$, in which case there is an unlimited number of solutions.

**Ex. 358.** Construct a circle of given radius, having its center on a given straight line and touching another given straight line. How many solutions may there be?

**Solution.** At any point $E$ in $CD$, one of the given lines, draw $EF \perp CD$ and equal to the given radius.



Through $F$ draw $FG \parallel CD$, cutting $AB$, the other given line, as in $O$.

With the given radius and $O$ as a center describe a ⊙.

Then, this ⊙ is the ⊙ required.

**Proof.** Const., § 151, every point in $FG$ is at the distance of the given radius from $CD$; hence, the ⊙ described with center $O$ and the given radius touches $CD$.

**Sch.** Since there may be a parallel on the other side of *CD*, there may be another solution.

If *CD* is parallel to *AB*, there is no solution unless the given radius is equal to the perpendicular distance between *CD* and *AB*, in which case there is an unlimited number of solutions.

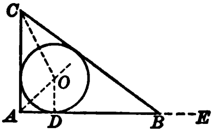
**Ex. 359.** Construct a right triangle, having given the radius of the inscribed circle and one of the sides containing the right angle.

**Solution.** Draw an indefinite line *AE*; at *A* draw $AC \perp AE$ and equal to the given side.

Bisect $\angle CAE$; on *AE* take $AD =$ the given radius, and at *D* erect a $\perp$ to *AE* meeting the bisector as in *O*.

Draw *CO*, and construct $\angle OCB = \angle OCA$, the side *CB* meeting *AE* as at *B*.

Then, $\triangle ABC$ is the $\triangle$ required.



**Proof.** § 134, every point in *AO* is equidistant from *AB* and *AC*, and every point in *CO* is equidistant from *AC* and *BC*; hence, the intersection *O* is at the distance *OD* from *AB*, *AC*, and *BC*, and *OD* is the radius of the inscribed ⊙.

Since $\angle OAD = \frac{1}{2}$ rt. $\angle$, § 111, $\angle AOD = \frac{1}{2}$ rt. $\angle$, and, § 118, $OD = AD =$ the given radius.

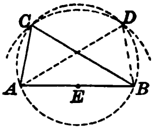
**Ex. 360.** Construct a triangle, having given the base, the vertical angle, and the length of the median to the base.

**Solution.** On $AB =$ the given base, describe a segment that will contain an $\angle =$ the given $\angle$.

Bisect *AB* and from *E* its middle point as center with a radius $=$ the given median, describe an arc intersecting the arc of the segment in *C* and *D*.

Draw *AC* and *BC*.

Then, $\triangle ABC$ is the $\triangle$ required.



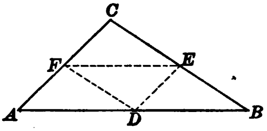
**Sch.** The $\triangle ABD$ also fulfills the required conditions.

If the given $\angle$ is a rt. $\angle$, there is an unlimited number of △s that would fulfill the conditions.

**Ex. 361.** Construct a triangle, having given the three middle points of its sides.

**Solution.** Join the given points *D*, *E*, and *F* by the lines *DE*, *FE*, and *FD*.

Through *F*, *D*, and *E* draw lines respectively parallel to *DE*, *FE*, and *FD* and produce them until they intersect as in *A*, *B*, and *C*.



Then, $\triangle ABC$ is the $\triangle$ required.

**Proof.** § 153, $AD = FE$, and $BD = FE$; hence, $AD = DB$, and $D$ is the middle point of $AB$.

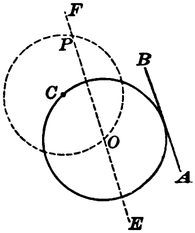
Similarly, $E$ and $F$ are the middle points of $BC$ and $AC$ respectively.

Ex. **362.** Construct a circle of given radius to pass through a given point and touch a given straight line.

**Solution.** Draw $EF \parallel AB$, the given line, at a distance from $AB$ equal to the given radius; with the given point $C$ as center and the given radius, describe a ⊙ intersecting $EF$ as in $O$ and $P$.

With $O$ as center and the given radius describe a ⊙.

Then, the ⊙ whose center is $O$ is the required ⊙.



**Sch.** Since there may be a parallel on each side of $AB$, theoretically there are four solutions, but only two are possible under any given conditions.

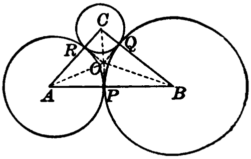
If $C$ is not in $AB$ and its distance from $AB$ is less than twice the given radius, the ⊙ of which it is the center will cut $EF$ in two points $O$ and $P$, but it cannot cut the other parallel; consequently there are only two solutions.

If the distance of $C$ from $AB$ = twice the given radius, the ⊙ whose center is $C$ will be tangent to $EF$ and there will be only one solution; if this distance is greater than twice the given radius, no solution is possible.

If $C$ is in $AB$, the ⊙ of which it is center will be tangent to both parallels and there will be two solutions.

Ex. **363.** From the vertices of a triangle as centers, to describe three circles which shall be tangent to each other.

**Solution.** Bisect ∠s $A$, $B$, and $C$ of the given $\triangle ABC$ and produce the bisectors to intersect as at $O$.



Draw $OP \perp AB$, $OQ \perp BC$, and $OR \perp AC$.

With $A$ as center and radius $AP$ describe a ⊙.

With $B$ as center and radius $BP$ describe a ⊙.

With $C$ as center and radius $CQ$ describe a ⊙.

Then, the Ⓢ whose centers are $A$, $B$, and $C$ are the Ⓢ required.

**Proof.** In rt. $\triangle$s $APO$ and $ARO$, § 169, $OP = OR$, and $AO$ is common; $\therefore$ § 123, $\triangle APO = \triangle ARO$, and $AP = AR$.

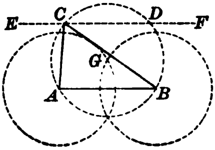
Similarly, $BP = BQ$, and $CR = CQ$.

Hence, the ⊙ whose center is $A$ passes through $P$ and $R$, the ⊙ whose center is $B$ passes through $P$ and $Q$, and the ⊙ whose center is $C$ passes through $R$ and $Q$.

Hence, § 205, $OP$ is a common tangent to the Ⓢ whose centers are $A$ and $B$; consequently, § 185, these Ⓢ are tangent to each other, and similarly, they are each tangent to the ⊙ whose center is $C$.

**Ex. 364.** Construct a triangle, having given the base, the altitude, and the radius of the circumscribed circle.

**Solution.** Draw $AB =$ the given base; draw $EF \parallel AB$ and at a distance from $AB$ equal to the given altitude.



With $A$ and $B$ as centers and the given radius describe Ⓢ intersecting as in $G$ on the same side of $AB$ as $EF$.

With $G$ as center and the same radius describe a ⊙ intersecting $EF$ in $C$ and $D$.

Draw $AC$ and $BC$.

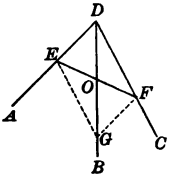
Then, $\triangle ABC$ is the $\triangle$ required.

**Proof.** By const., $AG = BG = CG =$ the given radius; hence, the ⊙ whose center is $G$ is circumscribed about $\triangle ABC$, whose altitude $=$ the distance between $AB$ and $EF =$ the given altitude and whose base $AB =$ the given base.

**Sch.** If we draw $AD$ and $BD$ another solution is obtained.

**Ex. 365.** Three given straight lines meet at a point; draw another straight line so that the two portions of it intercepted between the given lines are equal. How many solutions may there be?

**Solution.** Let $AD$, $BD$, and $CD$ be the given lines intersecting at $D$.



From any point of $AD$, as $E$, draw $EG \parallel DC$, meeting $BD$ as at $G$; draw $GF \parallel AD$, meeting $CD$ as in $F$.

Draw $EF$ cutting $BD$ as at $O$.

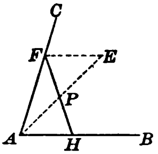
Then, $EF$ is the line required.

**Proof.** § 140, $EGFD$ is a ▱, and, § 154, $DG$ and $EF$ bisect each other at $O$; hence, $EO = OF$.

**Sch.** Since $E$ is any point of $AD$, there may be an unlimited number of solutions.

**Ex. 366.** Through a given point, between two intersecting straight lines, to draw a line terminated by the given lines and bisected at the given point.

**Solution.** Let $AB$ and $AC$ be the given lines and let $P$ be the given point.



Draw $AP$ and produce it to $E$, making $PE = AP$.

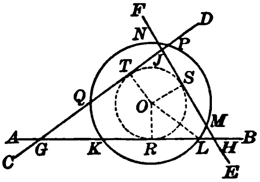
Draw $EF \parallel BA$ meeting $AC$ as in $F$.

Draw $FP$ and produce it to meet $AB$ as in $H$.

Then, $FH$ is the line required.

**Proof.** Const., $AP = PE$, § 59, $\angle APH = \angle EPF$, and, § 73, $\angle PAH = \angle PEF$; $\therefore$ § 102, $\triangle APH = \triangle EPF$, and $PH = PF$; that is, $FH$ is bisected at $P$.

**Ex. 367.** Construct a circle to intercept equal chords of given length on three given straight lines.



**Solution.** Place the given lines in the positions $AB$, $CD$, and $EF$ intersecting to form the $\triangle GHJ$.

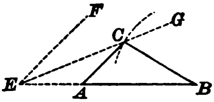
Draw the inscribed $\odot$ of $\triangle GHJ$, the center being $O$, and draw the radii $OR$, $OS$, and $OT$ to the points of tangency.

Take $RL = \frac{1}{2}$ the required length of chord, and with $O$ as center and $OL$ as radius describe a circumference intersecting $AB$, $CD$, and $EF$ in $K$ and $L$, $P$ and $Q$, and $M$ and $N$, respectively.

Then, $KL$, $PQ$, and $MN$ are the required chords.

**Proof.** § 205, $OR \perp KL$; hence, § 200, $R$ is the middle point of $KL$, and since, const., $RL = \frac{1}{2}$ the required length of chord, $KL =$ the required length, and since, Ax. 14, $OR = OT = OS$, § 202, $KL = PQ = MN$.

**Ex. 368.** Construct a triangle, having given one angle, the opposite side, and the sum of the other two sides.



**Solution.** Draw $EB =$ the given sum of two sides; construct $\angle FEB =$ the given $\angle$ and draw $EG$ to bisect it. With $B$ as center and radius $=$ the given side describe an arc cutting $EG$, as at $C$, and draw $CA \parallel FE$, meeting $EB$ at $A$. Draw $BC$.

Then, $\triangle ABC$ is the $\triangle$ required.

**Proof.** § 76, $\angle CAB = \angle FEB =$ the given $\angle$, and, const., $BC$ (the side opposite) $=$ the given side. § 73, const., $\angle ACE = \angle CEF = \angle AEC$;

$\therefore$ § 118, $AE = AC$; hence, $AC + AB = EB =$ the given sum of two sides.

**Ex. 370.** Find the locus of a point at a given distance from a given point.

**Solution.** Since, § 173, every point in the circumference of a $\odot$ is equally distant from the center, with the given point as center and radius $=$ the given radius describe a $\odot$.

Then, the circumference of this $\odot$ is the required locus.

**Ex. 371.** Find the locus of a point equidistant from two parallel straight lines.

**Solution.** Since a $\perp$ measures the distance between two parallel lines, the middle point of a $\perp$ between the given lines is equidistant from them.

Since, § 80, a line through this point, parallel to one of the given lines, is parallel to the other, § 151, every point in the line is equidistant from the given lines, and every point not in the line is at a greater distance from one of the given lines than from the other.

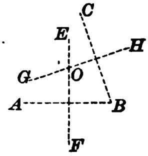
Hence, the locus of a point equidistant from two parallel straight lines is a line parallel to them and midway between them.

Ex. **372**. Find the locus of a point at a given distance from a given straight line.

**Solution.** Since, § 151, parallel lines are everywhere equally distant, the two lines (one on each side), parallel to the given line and at the given distance from it, constitute the required locus, for, § 151, every point in each is at the given distance from the given line, and every point not in either is at a distance, greater or less than the given distance, from the given line.

Ex. **373**. Find a point which is equidistant from three given points not in the same straight line.

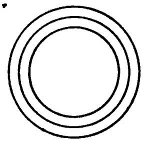
**Solution.** Since, § 104, every point equidistant from the extremities of a straight line lies in a ⊥ at its middle point, the required point is in the perpendicular bisector *FE* of the line joining two of the given points, *A* and *B*; it is also in the perpendicular bisector *HG* of the line *BC*, joining *B* with the other given point *C*.



Then, the intersection *O* of *FE* and *HG* is the required point.

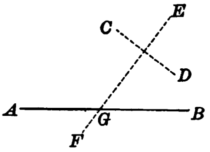
Ex. **374**. Find the locus of a point equidistant from the circumferences of two concentric circles.

**Solution.** Since the middle point of the part of a radius of the outer given circumference, intercepted between the two given circumferences, is equidistant from them, the required locus is the circumference of a ⊙ concentric with the given Ⓢ and having a radius equal to the radius of the smaller given ⊙ plus half the difference of the radii of the two given Ⓢ.



Ex. **375**. Find a point in a given straight line which is equidistant from two given points.

**Solution.** Since, § 104, every point equidistant from the extremities of a straight line lies in a ⊥ at its middle point, the required point is in the perpendicular bisector *EF* of the straight line *CD* joining the given points.

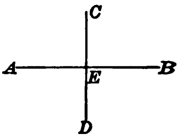


Hence, *G*, the intersection of *EF* and the given line *AB*, is the required point.

Ex. **376**. Find the locus of the center of a circle tangent to each of two parallel lines.

**Solution.** Since, § 173, the center must be equidistant from the given lines, its locus is the locus of a point equidistant from two parallel lines, which, Ex. 371, is a straight line parallel to the given lines and midway between them.

**Ex. 377.** Find the locus of the center of a circle which touches a given line at a given point.



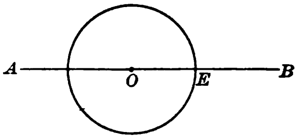
**Solution.** Since, § 205, a line $\perp$ a radius at its extremity is tangent to the ⊙, every ⊙ whose center is in *CD*, a line $\perp$ the given line *AB* at the given point *E*, and whose circumference passes through *E*, is tangent to *AB*, and since, § 51, there is only one $\perp$ to *AB* at *E*, any ⊙ whose center is not in *CD* is not tangent to *AB* at *E*.

Hence, *CD* is the required locus.

**Ex. 378.** Find the locus of the center of a circle of given radius when the circumference passes through a given point.

**Solution.** Since the center is at a distance from the given point = the given radius, its locus is the locus of a point at a given distance from a given point, which, Ex. 370, is the circumference of a ⊙ whose center is the given point and whose radius is the given radius.

**Ex. 379.** Find the locus of the center of a circle which is tangent to a given circle at a given point.



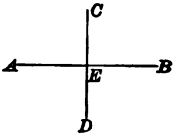
**Solution.** Since, § 213, the line of centers of any two tangent Ⓢ passes through the point of contact, the line *AB* through *O*, the center of the given ⊙, and *E*, the given point, will pass through the center of every ⊙ that is tangent to the given ⊙ at *E*; hence, *AB* is the required locus.

**Ex. 380.** Find the locus of the center of a circle of given radius and tangent to a given circle.

**Solution.** Since the center of the tangent ⊙ must be at the distance of the given radius from the circumference of the given ⊙, it will be on a circumference concentric with the circumference of the given ⊙ and having a radius equal to the sum or the difference of the given radius and the radius of the given ⊙, for if the center were at any point not in these circumferences, it would be at a distance greater or less than the given radius from the circumference of the given ⊙, and the circumference would not touch the given circumference or else it would cut it, and consequently would not be tangent to it.

Hence, these circumferences concentric with the given circumference constitute the required locus.

**Ex. 381.** Find the locus of the center of a circle passing through two given points.



**Solution.** Since, § 173, the center of a ⊙ passing through the given points *A* and *B* is equidistant from *A* and *B*, its locus is the locus of points equidistant from *A* and *B*; that is, § 104, *CD*, the perpendicular bisector of *AB* the line joining *A* and *B*, is the required locus.

**Ex. 382.** Find the locus of the center of a circle of given radius and tangent to a given line.

**Solution.** Since the center must be at a distance from the given line equal to the given radius, its locus is the locus of a point at a given distance from a given line; that is, Ex. 372, the two lines (one on each side) parallel to the given line and at a distance = the given radius from it constitute the required locus.

**Ex. 383.** Find the locus of the center of a circle tangent to each of two intersecting lines.

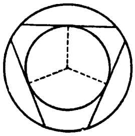
**Solution.** Since, § 173, the center must be equidistant from the given lines, its locus is the locus of a point which is equidistant from two intersecting lines; that is, Ex. 369, the bisectors of the two pairs of vertical ∠s formed by the lines constitute the required locus.

**Ex. 384.** Find the locus of the middle points of a system of parallel chords drawn in a circle.

**Solution.** Since, Ex. 187, a diameter that bisects one of the chords is perpendicular to it, § 72, this diameter is perpendicular to each chord of the system, and, § 200, passes through the middle point of each.

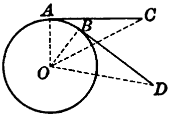
Hence, the diameter which bisects one of the chords is the required locus.

**Ex. 385.** Find the locus of the middle points of equal chords of a given circle.



**Solution.** Since, § 200, the ⊥s from the center of the ⊙ upon the chords meet them at their middle points, and since, the chords being equal, § 202, these ⊥s are equal, the middle points of the equal chords are equally distant from the center; hence, the circumference of a circle concentric with the given circle and having a radius = the distance to any of the equal chords is the required locus.

**Ex. 386.** Find the locus of the extremities of tangents of fixed length drawn to a given circle.

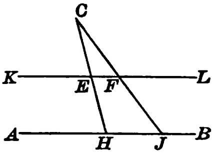


**Solution.** Let $O$ be the center of a given ⊙, $A$ and $B$ the points of contact of any two of the tangents, and $C$ and $D$ their extremities respectively. Draw $OA$, $OB$, $OC$, and $OD$.

Since, data, $AC = BD$, Ax. 14, $OA = OB$, and, §§ 205, 52, $\angle OAC = \angle OBD$, § 100, $\triangle OAC = \triangle OBD$, and $OC = OD$.

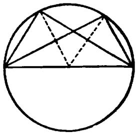
Hence, the extremity of each tangent is at the distance $OC$ from the center of the given ⊙; that is, Ex. 370, the circumference whose center is $O$ and radius $OC$ is the required locus.

Ex. **387**. Find the locus of the middle points of straight lines drawn from a given point to meet a given straight line.



**Solution.** Since any two straight lines from the given point *C*, as *CH* and *CJ*, meeting the given line *AB* in *H* and *J* respectively, form a △ *HJC*, § 159, the line *EF* connecting the middle points *E* and *F* of *CH* and *CJ* respectively, is parallel to *HJ* or *AB*; consequently, the line *KL* ∥ *AB* through *E* the middle point of one given line passes through *F* the middle point of any other given line and is the required locus.

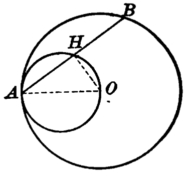
Ex. **388**. Find the locus of the vertex of a right triangle on a given base as hypotenuse.



**Solution.** Since, Ex. 163, the median to the hypotenuse of any rt. △ on the given base as hypotenuse = $\frac{1}{2}$ that base, the locus of the vertex is the locus of a point at a given distance from a given point, which, Ex. 370, is the circumference of a ⊙ whose center is the middle point of the given base, and whose radius is $\frac{1}{2}$ that base; that is, the circumference of the ⊙ on the given base as diameter is the required locus.

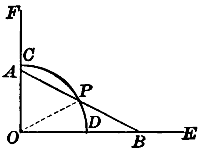
Ex. **389**. Find the locus of the middle points of all chords of a circle drawn from a fixed point in the circumference.

**Solution.** Draw any chord *AB* from the fixed point *A* in the circumference of the ⊙ whose center is *O*; draw *OH* ⊥ *AB*, and draw the radius *OA*.



Since, § 200, *H* is the middle point of *AB*, and, const., △ *OHA* is a rt. △ whose hypotenuse is *OA*, and since *AB* is *any* chord from *A*, the locus of the middle points of the chords is the locus of the vertex of a rt. △ on *OA* a given base as hypotenuse; that is, Ex. 388, the circumference of the ⊙, whose diameter is the radius *OA* of the given ⊙, is the required locus.

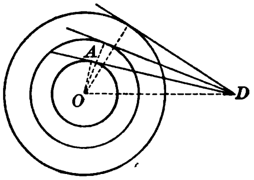
Ex. **390**. Find the locus of the middle point of a straight line moving between the sides of a right angle.



**Solution.** Since the given line *AB* in any of its positions between the sides of the rt. ∠ *EOF* is the hypotenuse of a rt. △ *AOB*, Ex. 163, the line *OP* from the vertex *O* to the middle point *P* of *AB* equals $\frac{1}{2}$ *AB*, and, Ex. 370, the point *P* lies in the circumference of a ⊙ whose center is *O* and radius $\frac{1}{2}$ *AB*; but, since *AB* moves only between the sides *OE* and *OF*, the locus of *P* is the quadrant *CD* of that circumference included between *OE* and *OF*.

Ex. **391.** Find the locus of the points of contact of tangents from a fixed point to a system of concentric circles.

**Solution.** Let $A$ be the point of contact of the tangent $DA$ from the fixed point $D$ to any one of the concentric Ⓢ whose center is $O$.

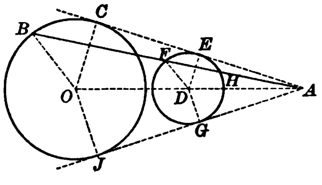


Draw the radius $OA$, and draw $OD$.

Since, § 205, ∠$OAD$ is a rt. ∠, the point of contact $A$ is at the vertex of a rt. △ on the base $OD$ as a hypotenuse; and, since $DA$ is any of the tangents, the locus of the points of contact is the locus of the vertex of a rt. △ on a given base as hypotenuse; that is, Ex. 388, the circumference of a ⊙, whose diameter is the line from the center $O$ to the fixed point $D$, is the required locus.

Ex. **392.** Find the locus of the middle points of secants drawn from a given point to a given circle.

**Solution.** Let $AB$ be any secant from the given point $A$ to the ⊙ whose center is $O$, and let $F$ be its middle point.



Draw $AO$ and $BO$.

Since $F$ is the middle point of $AB$, § 158, a line from $F$ drawn parallel to $BO$ meets $AO$ at its middle point $D$, and equals $\frac{1}{2}$ $OB$, the radius of the ⊙, and since $AB$ is any secant from $A$, the line drawn from the middle point of every secant in a similar way, meets $AO$ at $D$, and equals $\frac{1}{2}$ the radius; consequently, the middle points of the secants lie on a circumference whose center is $D$, and radius $\frac{1}{2}$ the radius of the given ⊙, and since there are secants only between the tangents $AJ$ and and $AC$, only the arc $GFE$ between $G$ and $E$, the middle points of these tangents, is the required locus.

**Cor.** *The arc GHE is the locus of the middle points of the external segments of the secants from A.*

## BOOK III

Ex. **393.** If the vertical angle of an isosceles triangle is 30°, what is its ratio to each of the base angles?

**Solution.** Each base ∠ $= \frac{1}{2}(180° - 30°) = 75°$; then, the required ratio is $30 : 75$, or $2 : 5$.

Ex. **394.** If the exterior angle at the base of an isosceles triangle is 100°, what is its ratio to each angle of the triangle?

**Solution.** The adj. int. ∠ $= 180° - 100° = 80°$; consequently, § 116, the other base ∠ $= 80°$, and the vert. ∠ $= 180° - (80° + 80°) = 20°$.

Then, the ratio of the ext. $\angle$ to each base $\angle$ is $100:80$, or $5:4$; and its ratio to the vert. $\angle$ is $100:20$, or $5:1$.

**Ex. 395.** If one of the acute angles of a right triangle is 40°, what is its ratio to the other acute angle? To the right angle?

**Solution.** § 111, the other acute $\angle = 90° - 40° = 50°$, and the ratio of the given $\angle$ to it is $40:50$, or $4:5$.

The ratio of the given $\angle$ to the rt. $\angle$ is $40:90$, or $4:9$.

**Ex. 396.** The interior angles on the same side of a transversal cutting two parallel lines are to each other as 3 to 2. How many degrees are there in each angle?

**Solution.** For every 3° in one of the given $\angle$s there are 2° in the other and 5° in their sum; but, § 78, the sum of the $\angle$s $= 180°$, hence, the larger $\angle = \frac{3}{5}$ of 180°, or 108°, and the other $\angle = \frac{2}{5}$ of 180° or 72°.

**Ex. 397.** The vertical angle of an isosceles triangle has the same ratio to a right angle that an angle of 40° has to an angle of an equilateral triangle. How many degrees are there in each angle of the isosceles triangle?

**Solution.** $\text{Vert.} \angle : 90° = 40° : 60°$;

whence, § 269, $\text{vert.} \angle = \dfrac{90 \times 40}{60} = 60°$.

Each base $\angle = \frac{1}{2}(180° - 60°) = 60°$.

**Ex. 398.** If $a:b = c:d$, prove that $a:a+b = c:c+d$.

**Proof.** Data, § 276, $a+b:a = c+d:c$; $\therefore$ § 275, $a:a+b = c:c+d$.

**Ex. 399.** If $a:b = b:c$, prove that $a+b:b+c = a:b$.

**Proof.** § 276, $a+b:b = b+c:c$; $\therefore$ § 274, $a+b:b+c = b:c$;

but, data, $a:b = b:c$; hence, $a+b:b+c = a:b$.

**Ex. 400.** *AD* bisects angle *A* at the base of the isosceles triangle *ABC*, and meets the side *BC* in *D*. If angle *C* is 68°, what is its ratio to angle *ADB*?

**Solution.** $\angle BAC = \frac{1}{2}(180° - 68°) = 56°$; hence, $\angle DAB = \frac{1}{2}$ of $56° = 28°$,

and since $\angle B = \angle BAC = 56°$, $\angle ADB = 180° - (56° + 28°) = 96°$.

Hence, $\angle C : \angle ADB = 68° : 96° = 17:24$.

**Ex. 401.** The sum of the angles of a polygon expressed in right angles is to the number of its sides as 4 is to 3. How many sides has the polygon?

**Solution.** § 166, the sum of $\angle$s of the polygon $= (n-2)\,2$ rt. $\angle$s;

then, data, $(n-2)\,2 : n = 4:3$;

whence, $n = 6$.

**Ex. 402.** If the angle formed by two secants intersecting without a circle is 30° and the smaller of the intercepted arcs is 20°, what is the ratio of the smaller arc to the larger?

**Solution.** Let $x$ denote the number of degrees in the larger arc.

Then, § 234, the given ∠ is measured by $\frac{1}{2}(x - 20°)$;

hence, $\frac{1}{2}(x - 20°) = 30°$,

from which $x = 80°$;

consequently, smaller arc : larger arc = 20° : 80° = 1 : 4.

**Ex. 403.** If the angle formed by two intersecting chords of a circle is 40° and one of the intercepted arcs is 30°, what is the ratio of that arc to the opposite intercepted arc?

**Solution.** Let $x$ denote the number of degrees in the opposite arc.

Then, § 230, the given ∠ is measured by $\frac{1}{2}(30° + x)$;

hence, $\frac{1}{2}(30° + x) = 40°$; whence $x = 50°$;

consequently, the given arc : opposite arc = 30° : 50° = 3 : 5.

**Ex. 404.** If $a : b = c : d$, prove that $2a + b : b = 2c + d : d$.

**Proof.** Data and § 276, $a + b : a = c + d : c$;

∴ § 276, $2a + b : a = 2c + d : c$; and, § 274, $2a + b : 2c + d = a : c$;

but, data, § 274, $a : c = b : d$; hence, $2a + b : 2c + d = b : d$,

and, § 274, $2a + b : b = 2c + d : d$.

**Ex. 405.** If $a : b = c : d$, prove that $a : 3a + b = c : 3c + d$.

**Proof.** Ex. 398, $a : a + b = c : c + d$, $a : 2a + b = c : 2c + d$,

and, $a : 3a + b = c : 3c + d$.

**Ex. 406.** If $a : b = b : c$, prove that $2a - b : a = 2b - c : b$.

**Proof.** § 277, $a - b : a = b - c : b$;

hence, § 276, $2a - b : a = 2b - c : b$.

**Ex. 407.** If $a : b = b : c$, prove that $a + 3b : b = b + 3c : c$.

**Proof.** § 276, $a + b : b = b + c : c$, $a + 2b : b = b + 2c : c$,

and $a + 3b : b = b + 3c : c$.

**Ex. 408.** Make the changes that may be made upon the following proportion without destroying the equality of the ratios: 16 : 36 = 4 : 9.

**Solution.** (1) § 274, 16 : 4 = 36 : 9.

(2) § 275, 36 : 16 = 9 : 4.

(3) § 276, 16 + 36 : 36 = 4 + 9 : 9; that is, 52 : 36 = 13 : 9,

or 16 + 36 : 16 = 4 + 9 : 4; that is, 52 : 16 = 13 : 4.

(4) § 277, 36 − 16 : 36 = 9 − 4 : 9; that is, 20 : 36 = 5 : 9,

or 36 − 16 : 16 = 9 − 4 : 4; that is, 20 : 16 = 5 : 4.

(5) § 278, $36+16:36-16=9+4:9-4$; that is, $52:20=13:5$.

(6) § 279, $16+4:36+9=16:36$; that is, $20:45=16:36$,

or $16+4:36+9=4:9$; that is, $20:45=4:9$.

(7) § 282, $2\times 16:2\times 36=4:9$; that is, $32:72=4:9$,

or $16:36=2\times 4:2\times 9$; that is, $16:36=8:18$.

(8) § 283, $16\div 2:36\div 2=4:9$; that is, $8:18=4:9$,

or $16:36=4\div 2:9\div 2$; that is, $16:36=2:4\frac{1}{2}$.

(9) § 284, $16\times 2:36=4\times 2:9$; that is, $32:36=8:9$,

or $16:36\times 2=4:9\times 2$; that is, $16:72=4:18$.

(10) § 285, $16\div 2:36=4\div 2:9$; that is, $8:36=2:9$,

or $16:36\div 2=4:9\div 2$; that is, $16:18=4:4\frac{1}{2}$.

(11) § 288, $16^2:36^2=4^2:9^2$; that is, $256:1296=16:81$,

or $\sqrt{16}:\sqrt{36}=\sqrt{4}:\sqrt{9}$; that is, $4:6=2:3$.

Ex. **409.** If $a:b=c:d$, prove that $ma:nb=mc:nd$.

**Proof.** Data, § 284, $ma:b=mc:d$, and $ma:nb=mc:nd$.

Ex. **410.** If $a:b=c:d$, prove that $a+4b:b=c+4d:d$.

**Proof.** Data, § 284, $a:4b=c:4d$;

∴ § 276, $a+4b:a=c+4d:c$;

hence, § 274, $a+4b:c+4d=a:c$;

but, data, § 274, $a:c=b:d$;

hence, $a+4b:c+4d=b:d$;

consequently, § 274, $a+4b:b=c+4d:d$.

Ex. **411.** If $a:b=b:c$, prove $a^2+ab:b^2+bc=a:c$.

**Proof.** Data, § 282, $a^2:ab=b^2:bc$;

∴ § 276, $a^2+ab:a^2=b^2+bc:b^2$,

and, § 274, $a^2+ab:b^2+bc=a^2:b^2$;

but, data and § 269, $b^2=ac$;

hence, substituting, $a^2+ab:b^2+bc=a^2:ac$;

consequently, § 283, $a^2+ab:b^2+bc=a:c$.

Ex. **412.** If $a:b=b:c$, prove that $a:c=(a+b)^2:(b+c)^2$.

**Proof.** Data, § 276, $a+b:a=b+c:b$,

and, § 274, $a+b:b+c=a:b$;

hence, § 288, $(a+b)^2:(b+c)^2=a^2:b^2$;

but, data and § 269, $b^2=ac$;

∴ substituting, $(a+b)^2:(b+c)^2=a^2:ac=a:c$;

that is, $a:c=(a+b)^2:(b+c)^2$.

**Ex. 413.** If $a : b = m : n$, and $b : c = n : o$, prove that $a : c = m : o$.

**Proof.** Data, $a : b = m : n$,

and $b : c = n : o$;

multiplying, § 287, $a : c = m : o$.

**Ex. 414.** If $a : b = c : d$, prove that

$$ma - nb : ma + nb = mc - nd : mc + nd.$$

**Proof.** Data, Ex. 409, $ma : nb = mc : nd$;

hence, § 278, $ma - nb : ma + nb = mc - nd : mc + nd$.

**Ex. 415.** If $a : b = c : d$, prove that

$$3a + 4b : 4a - 5b = 3c + 4d : 4c - 5d.$$

**Proof.** Data, Ex. 409, $3a : 4b = 3c : 4d$;

§ 276, $3a + 4b : 4b = 3c + 4d : 4d$,

and, § 285, $3a + 4b : b = 3c + 4d : d$,

or, § 274, $3a + 4b : 3c + 4d = b : d$.

Similarly it may be shown that $4a - 5b : 4c - 5d = b : d$.

Hence, $3a + 4b : 3c + 4d = 4a - 5b : 4c - 5d$;

that is, § 274, $3a + 4b : 4a - 5b = 3c + 4d : 4c - 5d$.

## BOOK IV

**Ex. 416.** The base of a triangle is 10 ft., and the other sides 8 ft. and 12 ft. Find the segments of the base made by the bisector of the vertical angle.

**Solution.** Let $x =$ number of feet in the segment adjacent to the 12 ft. side; then, $10 - x =$ number of feet in the other segment.

§ 292, $x : 10 - x = 12 : 8$;

whence, $8x = 120 - 12x$,

and $x = 6$ ft;

consequently, $10 - x = 4$ ft.

Hence, the segments required are 6 ft. and 4 ft.

**Ex. 417.** The sides $AC$ and $BC$ of the triangle $ABC$ are 5 ft. and 8 ft. respectively. If a line drawn parallel to the base divides $AC$ into segments of 2 ft. and 3 ft., what are the segments into which it divides $BC$?

**Solution.** Denote the segment of $BC$ corresponding to 2 by $x$.

Then, § 290, $5 : 2 = 8 : x$;

whence, § 269, $5x = 16$, and $x = 3\frac{1}{5}$;

consequently, one segment of $BC = 3\frac{1}{5}$ ft., and the other $= 8$ ft. $- 3\frac{1}{5}$ ft. $= 4\frac{4}{5}$ ft.

Ex. **418.** A line drawn parallel to the base of triangle $ABC$ divides $AC$ into segments of $3^{dm}$ and $8^{dm}$, respectively, and the segment of $BC$, corresponding to $3^{dm}$, is $5^{dm}$. What is the length of $BC$?

**Solution.** $BC - 5 =$ the other segment of $BC$.

Then, § 289, $BC - 5 : 5 = 8 : 3$;

∴ § 269, $3(BC - 5) = 40$;

whence, $BC = 18.33^{dm}+.$*

Ex. **419.** The sides $AB$, $BC$, and $AC$ of the triangle $ABC$ are respectively 8 in., 5 in., and 10 in. The bisector of the exterior angle at $C$ meets $AB$ produced in $E$. What is the length of $BE$?

**Solution.** § 293, $AE : BE = AC : BC$;

that is, $BE + 8 : BE = 10 : 5$;

∴ $5\,BE + 40 = 10\,BE$;

whence, $BE = 8$ in.

Ex. **420.** In triangle $ABC$, $AC$ is $15^{m}$ and $BC$ is $5^{m}$. The bisector of the exterior angle at $C$ meets $AB$ produced in $E$. If $AE$ is $21^{m}$, what is the length of the side $AB$?

**Solution.** § 293, $AE : BE = AC : BC$, or $21 : BE = 15 : 5$;

∴ § 269, $15\,BE = 105$; whence, $BE = 7$, and $AB = AE - BE = 14^{m}$.

Ex. **421.** The bisectors of an interior and exterior angle at $C$ of the triangle $ABC$ meet the opposite side and the opposite side produced in $E$ and $F$, respectively. If $AB$ is 14 in. and $EB$ is 4 in., what are the internal and external segments of $AB$?

**Solution.** § 298, $AE : EB = AF : BF$, or $AB - EB : EB = AB + BF : BF$;

that is, $10 : 4 = 14 + BF : BF$; ∴ § 269, $10\,BF = 56 + 4\,BF$; whence, $BF = 9\frac{1}{3}$;

consequently, $AF = 14 + 9\frac{1}{3} = 23\frac{1}{3}$. $AE = 14 - 4 = 10$.

That is, the required segments are 10 in., 4 in., $23\frac{1}{3}$ in., and $9\frac{1}{3}$ in., respectively.

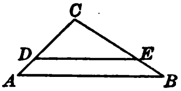
Ex. **422.** The sides of a triangle are 5, 7, and 9. If the side of a similar triangle homologous to 7 is 8, what are the other sides of the triangle?

**Solution.** Denote the sides homologous to 9 and 5 by $x$ and $y$ respectively.

Then, § 299, $x : 9 = 8 : 7$; whence, $x = \frac{9 \times 8}{7} = 10\frac{2}{7}$,

and $y : 5 = 8 : 7$; whence, $y = \frac{5 \times 8}{7} = 5\frac{5}{7}$.

**Ex. 423.** The sides of a triangle are $8^{dm}$, $10^{dm}$, and $12^{dm}$ in length respectively. If a line $9^{dm}$ long, parallel to the longest side, terminates in the other two, what are the segments into which it divides them?



**Solution.** Let the sides $AB$, $BC$, and $AC$ of $\triangle ABC$ equal $12^{dm}$, $10^{dm}$, and $8^{dm}$ respectively, and let $DE \parallel AB$ and terminated in $AC$ and $BC$ equal $9^{dm}$.

Then, §§ 290, 306, $\triangle$s $ABC$ and $DEC$ are similar, and, § 299, $AB : DE = BC : EC$; that is, $12 : 9 = 10 : EC$;

whence, $EC = \dfrac{9 \times 10}{12} = 7.5^{dm}$, and $BE = 10^{dm} - 7.5^{dm} = 2.5^{dm}$.

Also $AB : DE = AC : DC$; that is, $12 : 9 = 8 : DC$;

whence, $DC = \dfrac{9 \times 8}{12} = 6^{dm}$, and $AD = 8^{dm} - 6^{dm} = 2^{dm}$.

**Ex. 424.** If the bisector of an interior angle of a triangle divides the side opposite the angle into two segments which are 6 ft. and 8 ft. respectively, and if the side of the triangle adjacent to the 8 ft. segment is 20 ft., what is the length of the other side of the triangle?

**Solution.** Denote the required side by $x$.

Then, § 292, $8 : 6 = 20 : x$; $\therefore$ § 269, $8x = 120$, and $x = 15$ ft.

**Ex. 425.** If a stick 3 ft. long, in a vertical position, casts a shadow 1 ft. 7½ in. long, how high is a church steeple which at the same time casts a shadow 78 ft. in length?

**Solution.** 1 ft. 7½ in. $= \frac{39}{24}$ ft. Denote the height of the steeple by $x$.

Then, § 299, $x : 3 = 78 : \frac{39}{24}$; whence, $x = 144$ ft.

PROP. XVII, page 163.

**Proof.** §§ 290, 289, $(AG : AE) = AH : AD = HG : DE = GB : EF$;

but, const., $AD = l,\ DE = m$, and $EF = n$;

$\therefore$ substituting, $AH : l = HG : m = GB : n$.

PROP. XVIII, page 164.

**Proof.** § 289, $AE : EF = AD : DG$;

$\therefore$ substituting $l$, $m$, and $n$ for their equals $AE$, $AD$, and $EF$, respectively,

$l : n = m : DG$; that is, § 274, $l : m = n : DG$.

PROP. XIX, page 164.

**Proof.** § 289, $AE : EF = AD : DG$;

$\therefore$ substituting $l$ and $m$ for their equals $AE$, and $EF$ and $AD$ respectively,

$$l : m = m : DG.$$

PROP. XX, page 165.

**Proof.** § 314, $AC : CE = CE : CD$;

$\therefore$ substituting $l$ and $m$ for their equals $AC$ and $CD$ respectively,

$$l : CE = CE : m.$$

Ex. **426**. Three lines are $10^{cm}$, $12^{cm}$, and $16^{cm}$. Construct their fourth proportional.

**Solution.** Proceed as in § 317, making $l = 10^{cm}$, $m = 12^{cm}$, and $n = 16^{cm}$.

Ex. **427**. Two lines are $11^{cm}$ and $9^{cm}$. Construct their third proportional.

**Solution.** Proceed as in § 318, making $l = 11^{cm}$ and $m = 9^{cm}$.

Ex. **428**. Two lines are $6^{cm}$ and $2^{cm}$. Construct their mean proportional.

**Solution.** Proceed as in § 319, making $l = 2^{cm}$ and $m = 6^{cm}$.

Ex. **429**. Tangents are drawn through a point $6^m$ from the circumference of a circle whose radius is $9^m$. Find the length of the tangents.

**Solution.** From the given point draw a secant through the center of the ⊙; then, since the distance from the point to the circumference must be measured along this secant, its external segment is $6^m$, and its whole length is $9^m \times 2 + 6^m = 24^m$; denoting the tangent by $t$, § 315, $24 : t = t : 6$; $\therefore$ § 269, $t^2 = 144$, and $t = \sqrt{144} = 12^m$; § 209, the other tangent $= 12^m$.

Ex. **430**. If one side of a polygon is 2 ft. 6 in. long, what is the length of the corresponding side of a similar polygon, if their perimeters are respectively 15 ft. and 25 ft.?

**Solution.** § 311, $15 : 25 = 2\frac{1}{2} : x$ (the required side); whence, $x = 4\frac{1}{6}$ ft. $=$ 4 ft. 2 in.

Ex. **431**. The shortest distance from a given point to the circumference of a given circle is 2 ft. The length of a tangent from the same point to the circumference is 3 ft. Find the diameter of the circle.

**Solution.** From the given point draw the secant through the center of the ⊙; then, its internal segment is the diameter, and its external segment $=2$ ft.; denoting the diameter by $x$, § 315, $x+2 : 3 = 3 : 2$; whence, $x = 2\frac{1}{2}$ ft.

Ex. **432**. Five straight lines passing through the same point intercept segments, on one of two parallel lines, of $12^{dm}$, $20^{dm}$, $28^{dm}$, and $36^{dm}$. The segment of the other parallel corresponding to the $20^{dm}$ segment is $15^{dm}$. Find the other segments.

**Solution.** Denote the segments corresponding to the $12^{dm}$, $28^{dm}$, and $36^{dm}$ segments by $x$, $y$, and $z$, respectively.

Then, § 308, $x : 12 = 15 : 20$; whence, $x = 9^{dm}$,

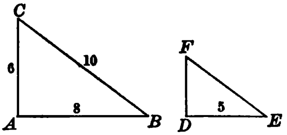
$15 : 20 = y : 28$; whence, $y = 21^{dm}$,

and $y : 28 = z : 36$, or $21 : 28 = z : 36$; whence, $z = 27^{dm}$.

PROP. XXII, p. 167.

**Proof.** Const., § 301, the corresponding ⧋ *ABC* and *FGH*, *ACD* and *FHJ*, *ADE* and *FJK* are similar; hence, § 309, *ABCDE* and *FGHJK* are similar.

**Ex. 433.** Construct a triangle whose sides are 6, 8, and 10; then, construct a similar triangle whose side homologous to 8 is 5.



**Solution.** By § 246, construct $\triangle ABC$ with sides 6, 8, and 10.

Draw $DE = 5$, and construct $\angle s\ D$ and $E$ equal respectively to $\angle s\ A$ and $B$, producing the sides to intersect as at $F$.

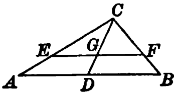
Then, $\triangle DEF$ is the required $\triangle$ similar to $\triangle ABC$.

**Proof.** Const. and § 301, $\triangle s\ DEF$ and $ABC$ are similar and fulfill the required conditions.

**Ex. 434.** Divide a line $10^{cm}$ long internally in extreme and mean ratio.

**Solution.** Proceed as in § 320, making $AB = 10^{cm}$.

**Ex. 435.** The median from the vertex of a triangle bisects every line drawn parallel to the base and terminated by the sides, or the sides produced.



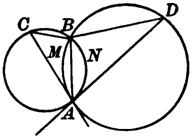
**Data:** $\triangle ABC$ and the median $CD$ to the base $AB$; also any line as $EF \parallel AB$, meeting the sides in $E$ and $F$, and cutting $CD$ in $G$.

**To prove** $EG = GF$.

**Proof.** § 308, $EG : AD = GF : DB$;

but, data, $AD = DB$; $\therefore$ § 272, $EG = GF$.

**Ex. 436.** Two circles intersect at $A$ and $B$, and at $A$ tangents are drawn, one to each circle, to meet the circumference of the other in $C$ and $D$ respectively; $BC$, $BD$, and $AB$ are drawn. Prove that $BD$ is a third proportional to $BC$ and $AB$.



**Proof.** In $\triangle s\ ABC$ and $ABD$,

§§ 231, 225, $\angle CAB = \angle ADB$ (each being measured by $\frac{1}{2}$ arc $AMB$),

also, $\angle ACB = \angle BAD$ (each being measured by $\frac{1}{2}$ arc $ANB$);

$\therefore$ § 301, $\triangle s\ ABC$ and $ABD$ are similar,

and, § 299, $BC : AB = AB : BD$;

that is, $BD$ is a third proportional to $BC$ and $AB$.

**Ex. 437.** The diameter $AB$, of a circle whose center is $O$, is divided at any point $C$, and $CD$ is drawn perpendicular to $AB$, meeting the circumference in $D$; $OD$ is drawn, and $CE$ perpendicular to $OD$. Prove that $DE$ is a third proportional to $AO$ and $DC$.

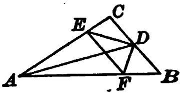
**Proof.** Data, $\triangle OCD$ is a rt. $\triangle$, and $CE \perp OD$;

$\therefore$ § 313, $OD : DC = DC : DE$;

but, Ax. 14, $OD = AO$; hence, $AO : DC = DC : DE$;

that is, $DE$ is a third proportional to $AO$ and $DC$.

**Ex. 438.** In the triangle $ABC$, $AD$ is the median to $BC$; the angles $ADC$ and $ADB$ are bisected by $DE$ and $DF$, meeting $AC$ and $AB$ in $E$ and $F$ respectively.



Then, $FE$ is parallel to $BC$.

**Proof.** In $\triangle ADC$, § 292, $AE : EC = AD : DC$,

and in $\triangle ABD$, $AF : FB = AD : DB$;

but, since, data, $DC = DB$, $AD : DC = AD : DB$;

$\therefore$ $AE : EC = AF : FB$; hence, § 291, $EF \parallel BC$.

**Ex. 439.** A secant from a given point without a circle is 1 ft. 6 in. long, and its external segment is 8 in. long. Find the length of a tangent to the circle from the same point.

**Solution.** Denote the tangent by $t$.

Then, § 315, $18 : t = t : 8$; whence, $t^2 = 144$, and $t = 12$ in.

**Ex. 440.** The radius of a circle is 6 in. What is the length of the tangents drawn from a point 12 in. from the center?

**Solution.** From the given point draw a secant through the center. Then, the secant = 12 in. + 6 in. = 18 in., and its external segment = 12 in. − 6 in. = 6 in.; denoting the tangent by $t$, § 315, $18 : t = t : 6$; whence, $t^2 = 108$, and $t = 10.39+$ in.

**Ex. 441.** If the tangent to a circle from a given point is $2^{\text{m}}$, and the radius of the circle is $15^{\text{dm}}$, find the distance from the point to the circumference.

**Solution.** From the given point draw a secant through the center. Then, its external segment = the required distance. Denote it by $x$; then, the secant $= x + 2 \cdot 15^{\text{dm}} = x + 30^{\text{dm}}$, and since the tangent $= 2^{\text{m}} = 20^{\text{dm}}$,

§ 315, $x + 30 : 20 = 20 : x$; whence, $x^2 + 30x = 400$, and $x = 10^{\text{dm}}$.

**Ex. 442.** If from the vertex $D$ of the parallelogram $ABCD$ a straight line is drawn, cutting $AB$ at $E$ and $CB$ produced at $F$, prove that $CF$ is a fourth proportional to $AE$, $AD$, and $AB$.

**Proof.** § 301, $\triangle$s $AED$ and $DFC$ are similar;

hence, § 299, $AE : AD = DC : CF$;

but, § 153, $DC = AB$; $\therefore$ $AE : AD = AB : CF$;

that is, $CF$ is a fourth proportional to $AE$, $AD$, and $AB$.

**Ex. 443.** If the segments of the hypotenuse of a right triangle made by the perpendicular from the vertex of the right angle are 6 in. and 4 ft., find the length of the perpendicular and the length of each of the sides about the right angle.

**Solution.** Let $CD$ be the $\perp$ from the vertex $C$ of the rt. $\angle$ of rt. $\triangle ABC$ to the base $AB$, and let $AD = 6$ in., or $\frac{1}{2}$ ft., and $BD = 4$ ft.

Then, § 312, $AD : CD = CD : BD$, or $\frac{1}{2} : CD = CD : 4$;

whence, $\overline{CD}^2 = 2$, and $CD = \sqrt{2} = 1.4142+$ ft.

§ 313, $AB : AC = AC : AD$, or $4\frac{1}{2} : AC = AC : \frac{1}{2}$;

whence, $\overline{AC}^2 = \frac{9}{4}$, and $AC = \frac{3}{2}$, or $1\frac{1}{2}$ ft.

Similarly, $\overline{BC}^2 = 4\frac{1}{2} \times 4 = 18$, and $BC = 3\sqrt{2} = 4.2426+$ ft.

**Ex. 444.** Find the length of the longest and of the shortest chord that can be drawn through a point $7\frac{1}{2}$ in. from the center of a circle whose radius is $19\frac{1}{2}$ in.

**Solution.** Let $P$ be the given point in the circle whose center is $O$. Draw $AB$ the diameter through $P$, also the chord $CD$ through $P$ and $\perp AB$.

Then, § 192, $AB$ is longest and, Ex. 195, $CD$ is the shortest chord through $P$.

$$AB = 19\tfrac{1}{2} \text{ in.} \times 2 = 39 \text{ in.}$$

§ 314, $AP : CP = CP : BP$,

or $19\frac{1}{2} + 7\frac{1}{2} : CP = CP : 19\frac{1}{2} - 7\frac{1}{2}$; that is, $27 : CP = CP : 12$;

whence, $\overline{CP}^2 = 324$, and $CP = 18$ in.;

but, § 200, $CP = \frac{1}{2} CD$; hence, $CD = 18$ in. $\times 2 = 36$ in.

**Ex. 445.** If the greater segment of a line divided internally in extreme and mean ratio is 36 in., what is the length of the line?

**Solution.** (See Fig. 1, § 295.) Let $AB$ be the line and $AC$ the greater segment.

Then, § 296, $AB : AC = AC : CB$,

or $AB : 36 = 36 : AB - 36$;

∴ § 269, $\overline{AB}^2 - 36\,AB = 1296$; whence, $AB = 58.24+$ in.

**Ex. 446.** The shorter segment of a line divided externally in extreme and mean ratio is $240^{dm}$. Find the length of the greater segment in meters.

**Solution.** (See Fig. 2, § 295.) Let $AB$ be divided externally at $E$ in extreme and mean ratio, $AE\,(= 240^{dm}$, or $24^{m})$ being the shorter segment.

Then, § 296, $AB : AE = AE : BE$,

or $BE - 24 : 24 = 24 : BE$;

∴ § 269, $\overline{BE}^2 - 24\,BE = 576$; whence, $BE = 38.83+^{m}$.

**Ex. 447.** Find the shorter segment of a line $12^{dm}$ long (1) when it is divided internally in extreme and mean ratio. (2) When it is divided externally in extreme and mean ratio.

**Solution.** Let $AB$ be divided internally at $C$ and externally at $D$ in extreme and mean ratio, $AC$ and $BD$ being the shorter segments respectively.

(1) § 298, $AB : BC = BC : AC$,

or $12 : BC = BC : 12 - BC$;

§ 269, $\overline{BC}^2 = 144 - 12\,BC$; whence, $BC = 7.41+^{\text{dm}}$,

and $AC = AB - BC = 12^{\text{dm}} - 7.41+^{\text{dm}} = 4.59-^{\text{dm}}$.

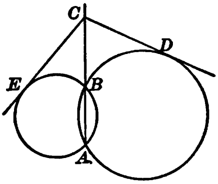
(2) $AB : BD = BD : AD$, or $12 : BD = BD : 12 + BD$;

∴ § 269, $\overline{BD}^2 = 144 + 12\,BD$; whence, $BD = 19.41+^{\text{dm}}$,

and $AD = AB + BD = 12^{\text{dm}} + 19.41+^{\text{dm}} = 31.41+^{\text{dm}}$.

Ex. **448**. The tangents to two intersecting circles drawn from any point in their common chord produced are equal.

**Data**: Two intersecting circles and their common chord $AB$; also two tangents $CE$ and $CD$, one to each circle, from $C$, a point in $AB$ produced.



**To prove** $CE = CD$.

**Proof.** § 315, $AC : CE = CE : BC$;

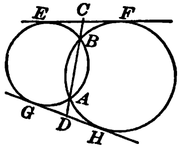
∴ § 269, $\overline{CE}^2 = AC \times BC$.

Similarly, $\overline{CD}^2 = AC \times BC$;

hence, $\overline{CE}^2 = \overline{CD}^2$, and $CE = CD$.

Ex. **449**. If the common chord of two intersecting circles is produced, it will bisect their common tangents.

**Data**: Two intersecting circles whose common chord $AB$ produced intersects their common tangents $EF$ and $GH$ in $C$ and $D$ respectively.



**To prove** $CE = CF$, and $DG = DH$.

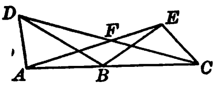
**Proof.** Since $C$ is in $AB$ produced and $CE$ and $CF$ are tangents respectively to the two circles,

Ex. 448, $CE = CF$.

Similarly, $DG = DH$.

Ex. **450**. $ABC$ is a straight line, $ABD$ and $BCE$ are triangles on the same side of it, having angle $ABD$ equal to angle $CBE$ and $AB : BC = BE : BD$.

If $AE$ and $CD$ intersect in $F$, triangle $AFC$ is isosceles.



**Proof.** Data, $\angle ABD = \angle CBE$;

∴ Ax. 2, $\angle ABD + \angle DBE = \angle CBE + \angle DBE$,

or $\angle ABE = \angle DBC$,

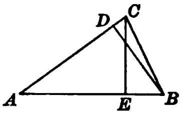
and, data, $AB : BC = BE : BD$;

∴ § 306, $\triangle$s $ABE$ and $DBC$ are similar;

hence, § 299, $\angle FAC = \angle FCA$; ∴ § 118, $\triangle AFC$ is isosceles.

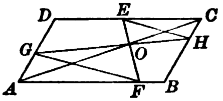
Ex. **451.** If in the triangle $ABC$, $CE$ and $BD$ are drawn perpendicular to the sides $AB$ and $AC$ respectively, these sides are reciprocally proportional to the perpendiculars upon them; that is,

$$AB : AC = BD : CE.$$



**Proof.** In the rt. $\triangle ABD$ and $AEC$, $\angle A$ is common;
hence, § 302, $\triangle ABD$ and $AEC$ are similar;
∴ § 299, $AB : AC = BD : CE.$

Ex. **452.** $ABCD$ is a parallelogram. If through $O$, any point in the diagonal $AC$, $EF$ and $GH$ are drawn, terminating in $DC$ and $AB$, and in $AD$ and $BC$, respectively, $EH$ is parallel to $GF$.

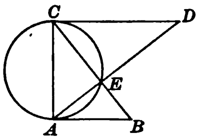


**Proof.** § 301, $\triangle AOG$ and $COH$ are similar;
∴ § 299, $OG : OH = OA : OC;$
similarly, $\triangle AOF$ and $EOC$ are similar,
and $OF : OE = OA : OC;$
hence, $OG : OH = OF : OE,$
and, § 59, $\angle GOF = \angle EOH;$
∴ § 306, $\triangle GOF$ and $EOH$ are similar, and, § 299, $\angle OFG = \angle OEH.$
Hence, § 75, $EH \parallel GF.$

Ex. **453.** Lines are drawn from a point $P$ to the vertices of the triangle $ABC$; through $D$, any point in $PA$, a line is drawn parallel to $AB$, meeting $PB$ at $E$, and through $E$, a line parallel to $BC$, meeting $PC$ at $F$. If $FD$ is drawn, triangle $DEF$ is similar to triangle $ABC$.

**Proof.** Since $DE \parallel AB$, § 290, $AP : DP = BP : EP,$
and since $EF \parallel BC$, $CP : FP = BP : EP;$
∴ $AP : DP = CP : FP$; hence, § 291, $DF \parallel AC;$
consequently, § 307, $\triangle ABC$ and $DEF$ are similar.

Ex. **454.** If two lines are tangent to a circle at the extremities of a diameter, and from the points of contact secants are drawn terminated respectively by the opposite tangent and intersecting the circumference at the same point, the diameter is a mean proportional between the tangents.



**Data:** $AB$ and $CD$ tangents at the extremities of diameter $AC$, and the secants $CB$ and $AD$ from the points of contact intersecting the circumference at $E$ and the tangents at $B$ and $D$ respectively.

**To prove** $AB : AC = AC : CD.$

**Proof.** § 54, $\angle EAB = \angle ADC$ (each being comp. of $\angle CAE$),
and, §§ 231, 225, $\angle EAB = \angle ACB$ (each being measured by $\frac{1}{2}$ arc $AE$);
∴ $\angle ADC = \angle ACB$, and, § 302, $\triangle ABC$ and $ADC$ are similar;
hence, § 299, $AB : AC = AC : CD.$

Ex. **455**. $AB$ and $AC$ are secants of a circle from the common point $A$, cutting the circumference in $D$ and $E$ respectively. Then, the secants are reciprocally proportional to their external segments; that is,

$$AB : AC = AE : AD.$$

**Proof.** Draw $CD$ and $BE$.

In $\triangle ABE$ and $ACD$, $\angle A$ is common,

and, § 225, $\angle B = \angle C$ (each being measured by $\frac{1}{2}$ arc $DE$);

∴ § 301, $\triangle ABE$ and $ACD$ are similar;

hence, § 299, $AB : AC = AE : AD.$

Ex. **456**. $AB$ and $CD$ are two chords of a circle intersecting at $E$.

Prove that $AE : DE = CE : BE.$

**Proof.** Draw $AD$ and $BC$.

§ 59, $\angle AED = \angle CEB,$

and, § 225, $\angle ADE = \angle EBC$ (each being measured by $\frac{1}{2}$ arc $AC$);

∴ § 301, $\triangle AED$ and $CEB$ are similar;

hence, § 299, $AE : DE = CE : BE.$

Ex. **457**. Two secants intersect without a circle. The segments of one are 4 ft. and 20 ft., and the external segment of the second is 16 ft. Find the length of the second secant.

**Solution.** Let $x$ denote the number of ft. in the length of the required secant.

Then, Ex. 455, $x : 20 + 4 = 20 : 16;$

∴ § 269, $16x = 20(20 + 4)$; whence, $x = 30$ ft.

Ex. **458**. From a point without a circle two secants are drawn, whose external segments are respectively $7^{\text{dm}}$ and $9^{\text{dm}}$, the internal segment of the latter being $13^{\text{dm}}$. What is the length of the first secant?

**Solution.** Let $x$ denote the length of the required secant in decimeters.

Then, Ex. 455, $x : 13 + 9 = 9 : 7;$

∴ $7x = 9(13 + 9)$; whence, $x = 28\frac{2}{7}$.

Ex. **459**. The segments of a chord intersected by another chord are 7 in. and 9 in., and one segment of the latter is 3 in. What is the other segment?

**Solution.** Let $x$ denote the length of the required segment in inches.

Then, Ex. 456, $x : 9 = 7 : 3;$

∴ $3x = 63$, and $x = 21$ in.

Ex. **460**. Two secants from the same point without a circle are $24^{\text{dm}}$ and $32^{\text{dm}}$ long. If the external segment of the less is $5^{\text{dm}}$, what is the external segment of the greater?

**Solution.** Let $x$ denote the length of the required segment in decimeters.

Then, Ex. 455, $24 : 32 = x : 5;$

∴ $32x = 120$, and $x = 3.75^{\text{dm}}$.

**Ex. 461.** Through a point $7^m$ from the circumference of a circle a secant $28^m$ long is drawn. If the internal segment of this secant is $17^m$, what is the radius of the circle?

**Solution.** From the given point draw a secant through the center of the circle. Then its external segment is $7^m$. Denote its length in meters by $x$.

Ex. 455, $x : 28 = (28 - 17) : 7$;

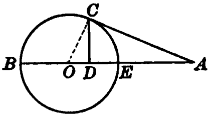
$\therefore$ $7x = 28(28 - 17)$;

whence, $x = 44^m$,

and the diameter = the internal segment of $x = 44^m - 7^m = 37^m$;

hence, the radius = $\frac{1}{2}$ of $37^m$, or $18.5^m$.

Ex. **462.** If from any point in the diameter of a circle produced a tangent is drawn, and a perpendicular from the point of contact is let fall on the diameter, the distances from the point without the circle to the foot of the perpendicular, the center of the circle, and the extremities of the diameter are in proportion.



**Data:** Circle whose center is $O$; a tangent to it, as $AC$, from $A$ a point on the diameter $BE$ produced; and the $\perp$ $CD$ from the point of contact to $BE$.

**To prove** distances $AD$, $AO$, $AB$, and $AE$ in proportion.

**Proof.** Draw $OC$.

Then, § 315, $AB : AC = AC : AE$, or $AB \times AE = \overline{AC}^2$,

and, § 313, $AO : AC = AC : AD$, or $AO \times AD = \overline{AC}^2$;

$\therefore$ $AB \times AE = AO \times AD$;

hence, § 273, $AB : AO = AD : AE$.

Ex. **463.** If the sides of a triangle are respectively $1.5^{Dm}$, $.12^{Hm}$, and $10^m$ long, what are the segments into which each side is divided by the bisector of the opposite angle?

**Solution.** Let $ABC$ be a $\triangle$ whose sides $AB$, $AC$, and $BC$ are $1.5^{Dm}$ (or $15^m$), $.12^{Hm}$ (or $12^m$), and $10^m$, respectively, and divided at the points $D$, $F$, and $E$, respectively, by the bisectors of the opposite $\angle$s.

(1) § 292, $AD : DB = AC : BC$;

$\therefore$ § 276, $AD + DB : DB = AC + BC : BC$;

that is, $15 : DB = 12 + 10 : 10$;

whence, $DB = \dfrac{15 \times 10}{12 + 10} = 6.818+^m$,

and $AD = 15^m - 6.818+^m = 8.182-^m$.

(2) Similarly, $AF + FC : FC = AB + BC : BC$;

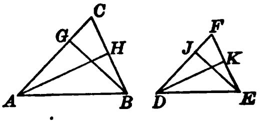
that is, $12 : FC = 15 + 10 = 10$;

whence, $FC = \dfrac{12 \times 10}{15 + 10} = 4.8^m$,

and $AF = 12^m - 4.8^m = 7.2^m$.

(3) Similarly, $BE + EC : EC = AB + AC : AC$;
that is, $10 : EC = 15 + 12 : 12$;
whence, $EC = \frac{10 \times 12}{15 + 12} = 4.44+^{\text{m}}$,
and $BE = 10^{\text{m}} - 4.44+^{\text{m}} = 5.56-^{\text{m}}$.

Ex. **464**. If an angle of one triangle is equal to an angle of another, and the perpendiculars from the vertices of the remaining angles to the sides opposite are proportional, the triangles are similar.

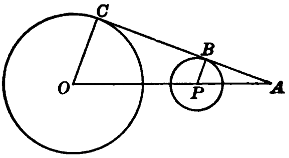


**Data**: $\triangle ABC$ and $DEF$, in which $\angle C = \angle F$, and the $\perp$s $AH$, $BG$, $DK$, and $EJ$ are proportional.

**To prove** $\triangle ABC$ and $DEF$ similar.

**Proof.** Data, $\angle C = \angle F$;
$\therefore$ § 302, rt. $\triangle CBG$ and $FEJ$ are similar,
and, § 299, $BG : EJ = BC : EF$;
similarly, $AH : DK = AC : DF$;
but, data, $AH : DK = BG : EJ$;
hence, $AC : DF = BC : EF$,
and since, $\angle C = \angle F$,
§ 306, $\triangle ABC$ and $DEF$ are similar.

Ex. **465**. If two circles are respectively 6 in. and 3 in. in diameter, and their centers are 10 in. apart, find the distance from the center of the smaller one to the point of intersection of their common exterior tangent with their line of centers produced.



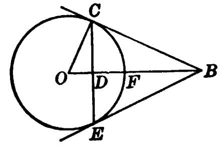
**Solution.** § 205, $\triangle ACO$ and $ABP$ are rt. $\triangle$,
and since $\angle A$ is common,
§ 302, $\triangle ACO$ and $ABP$ are similar;
hence, $AO : AP = OC : PB$;
that is, $10 + AP : AP = 3 : 1\frac{1}{2}$;
whence, $3\,AP = \frac{3}{2}(10 + AP)$,
$6\,AP = 30 + 3\,AP$,
and $AP = 10$ in.

Ex. **466**. Two intersecting chords of a circle are 38 ft. and 34 ft., respectively; the segments of the first are 8 ft. and 30 ft. Find the segments of the second.

**Solution.** Let $x$ denote the longer of the required segments; then, the shorter segment $= 34 - x$.

Ex. 456, $30 : x = 34 - x : 8$; $\therefore 34x - x^2 = 240$;
whence, $x = 24$ ft., and $34 - x = 10$ ft.

Ex. **467**. What is the length of a chord joining the points of contact of the tangents drawn from a point 13 in. from the center of a circle whose radius is 5 in.?



**Solution.** Since, § 209, Ax. 14, $B$ and $O$ are each equidistant from $C$ and $E$, § 106, $OB \perp CE$ at its middle point $D$; $\therefore$ § 313, $OD : OC = OC : OB$; that is, $OD : 5 = 5 : 13$;

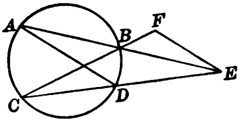
whence, $$OD = \frac{5 \times 5}{13} = \frac{25}{13} \text{ in.}$$

§ 312, $OD : CD = CD : BD$; that is, $\frac{25}{13} : CD = CD : (13 - \frac{25}{13})$;

whence, $$\overline{CD}^2 = \frac{25}{13} \times \frac{144}{13} = \frac{144 \times 25}{13^2}, \text{ and } CD = \frac{60}{13} = 4\tfrac{8}{13} \text{ in.},$$

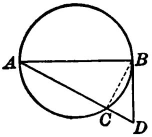
and $$CE = 2\,CD = 9\tfrac{8}{13} \text{ in.}$$

Ex. **468**. Chords $AB$ and $CD$ of a circle are produced in the direction of $B$ and $D$ respectively to meet in the point $E$, and through $E$ the line $EF$ is drawn parallel to $AD$ to meet $CB$ produced in $F$. Prove that $EF$ is a mean proportional between $FB$ and $FC$.



**Proof.** § 73, $\angle BEF = \angle BAD$, and, § 225, $\angle BCD = \angle BAD$ (each being measured by $\frac{1}{2}$ arc $BD$); hence, $\angle BEF = \angle BCD$, and, since $\angle F$ is common to $\triangle BEF$ and $CEF$, § 301, the $\triangle$ are similar; consequently, § 299, $FB : EF = EF : FC$.

Ex. **469**. $AB$ is a diameter of a circle, and through $A$ any straight line is drawn to cut the circumference in $C$ and the tangent at $B$ in $D$. Prove that $AC$ is a third proportional to $AD$ and $AB$.



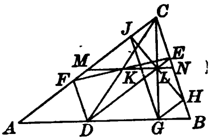
**Proof.** Draw $BC$. § 205, $\angle ABD$ is a rt. $\angle$, and, § 227, $\angle ACB$ is a rt. $\angle$.

Then, in $\triangle ABD$ and $ACB$, $\angle A$ is common;

$\therefore$ § 302, $\triangle ABD$ and $ACB$ are similar;

hence, § 299, $AD : AB = AB : AC$.

Ex. **470**. From any point in the base of a triangle straight lines are drawn parallel to the sides. Prove that the intersection of the diagonals of every parallelogram so formed lies in a line parallel to the base of the triangle.



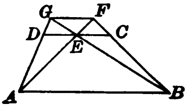
**Data:** $\triangle ABC$ and any points $D$ and $G$ in the base $AB$ from which are drawn lines $DE$ and $GH \parallel AC$, and $DF$ and $GJ \parallel BC$ forming ▱ $DECF$ and $GHCJ$ whose diagonals $FE$, $DC$, and $GC$, $HJ$, respectively, intersect in $K$ and $L$, respectively.

**To prove** $K$ and $L$ lie in a line parallel to $AB$.

**Proof.** Through $K$ and $L$ draw the line $MN$ meeting $AC$ and $BC$ in $M$ and $N$, respectively.

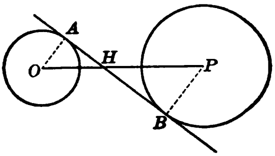
§ 154, $K$ is the middle point of $DC$, and $L$ is the middle point of $GC$; hence, in $\triangle DGC$, § 159, $KL \parallel DG$, or $MN \parallel AB$; that is, $K$ and $L$ lie in a line parallel to $AB$.

Ex. **471**. If $E$ is the middle point of one of the parallel sides $DC$ of the trapezoid $ABCD$, and $AE$ and $BE$ produced meet $BC$ and $AD$ produced in $F$ and $G$, respectively, prove that $GF$ is parallel to $AB$.



**Proof.** § 301, $\triangle$s $ABG$ and $DEG$ are similar;
∴ § 299, $AB : DE = BG : EG$;
similarly, $AB : EC = AF : EF$;
but, data, $DE = EC$; hence, $BG : EG = AF : EF$;
∴ § 277, $BG - EG : EG = AF - EF : EF$; that is, $BE : EG = AE : EF$,
and since, § 59, $\angle AEB = \angle GEF$, § 306, $\triangle$s $ABE$ and $GFE$ are similar,
and, § 299, $\angle EAB = \angle GFE$; hence, § 75, $GF \parallel AB$.

Ex. **472**. If a line tangent to two circles cuts their line of centers, the segments of the latter are to each other as the diameters of the circles.



**Data:** Circles whose centers are $O$ and $P$, and their common tangent $AB$ cutting their line of centers $OP$ at $H$.

**To prove**

$OH : HP =$ diameter of $\odot O$: diameter of $\odot P$.

**Proof.** Draw radii $OA$ and $PB$ to the points of contact.

Then, § 205, $\triangle$s $OAH$ and $PBH$ are rt. $\angle$s, and, § 59, $\angle OHA = \angle PHB$;
∴ § 302, $\triangle$s $OHA$ and $PHB$ are similar,
and, § 299, $OH : HP = OA : PB$, or, § 282, $OH : HP = 2\,OA : 2\,PB$;
that is, $OH : HP =$ diameter of $\odot O$ : diameter of $\odot P$.

Ex. **473**. The bisector of the vertical angle $C$ of the inscribed triangle $ABC$ cuts the base at $D$ and meets the circumference in $E$. Prove that $AC : CD = CE : BC$.

**Proof.** Draw $EB$. Data, $\angle ACD = \angle ECB$,
and, § 225, $\angle A = \angle E$ (each being measured by $\frac{1}{2}$ arc $BC$);
hence, § 301, $\triangle$s $ADC$ and $EBC$ are similar,
and, § 299, $AC : CD = CE : BC$.

Ex. **474**. Through any point $A$ of the circumference of a circle a tangent is drawn, and from $A$ two chords, $AB$ and $AC$; the chord $FG$ parallel to the tangent cuts $AB$ and $AC$ in $D$ and $E$ respectively.

Prove $AB : AE = AC : AD$.

**Proof.** Draw $BC$.

Then, in $\triangle$s $ABC$ and $AED$, $\angle A$ is common,

§ 225, $\angle ABC$ is measured by $\frac{1}{2}$ (arc $CG$ + arc $AG$),

and, § 230, $\angle AED$ is measured by $\frac{1}{2}$ (arc $CG$ + arc $AF$);

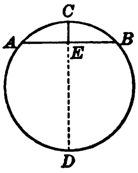
but, § 206, arc $AF$ = arc $AG$;

hence, $\angle ABC = \angle AED$;

∴ § 301, $\triangle$s $ABC$ and $AED$ are similar,

and, § 299, $AB : AE = AC : AD$.

Ex. **475**. The greatest distance of a chord 8 ft. in length from its arc is 4 in. Find the diameter of the circle.

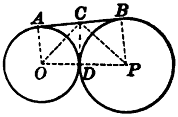


**Solution.** Let chord $AB = 8$ ft. and let $CD$ be the diameter $\perp AB$ intersecting it at $E$; then $CE$, the greatest distance of chord $AB$ from its arc = 4 in. = $\frac{1}{3}$ ft.

§ 314, $DE : AE = AE : CE$; that is, § 200, $DE : 4 = 4 : \frac{1}{3}$; whence,

$$DE = \frac{4 \times 4}{\frac{1}{3}} = 48 \text{ ft., and } CD = DE + CE = 48 \text{ ft.} + \tfrac{1}{3} \text{ ft.}$$
$$= 48\tfrac{1}{3} \text{ ft.}$$

Ex. **476**. If two circles are tangent externally, their common exterior tangent is a mean proportional between the diameters of the circles.



**Data**: Circles whose centers are $O$ and $P$ tangent to each other at $D$; also their common exterior tangent $AB$.

**To prove** diam. $\odot O : AB = AB :$ diam. $\odot P$.

**Proof.** Draw the common interior tangent $DC$ to meet $AB$ as at $C$; draw $OA$, $PB$, $OC$, and $PC$; also, § 213, draw $ODP$.

Then, Ex. 221, $OC$ bisects $\angle ACD$, and $PC$ bisects $\angle BCD$;

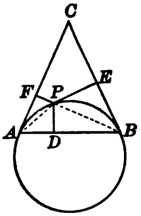
∴ Ex. 14, $\angle OCP$ is a rt. $\angle$, and since, § 205, $CD \perp OP$,

§ 312, $OD : CD = CD : DP$; but, § 209, $CD = AC = CB = \frac{1}{2} AB$;

hence, $\frac{1}{2}$ diam. $\odot O : \frac{1}{2} AB = \frac{1}{2} AB : \frac{1}{2}$ diam. $\odot P$;

that is, diam. $\odot O : AB = AB :$ diam. $\odot P$.

Ex. **477**. The perpendicular from any point of a circumference upon a chord is a mean proportional between the perpendiculars from the same point upon the tangents drawn at the extremities of the chord.



**Data**: Chord $AB$; tangents $AC$ and $BC$; any point $P$ on the circumference; and $PD$, $PF$, and $PE \perp$ respectively to $AB$, $AC$, and $BC$.

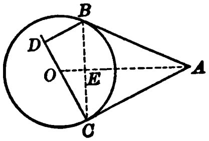
**To prove** $PF : PD = PD : PE$.

**Proof.** Draw $AP$ and $BP$.

§§ 231, 225, $\angle FAP = \angle DBP$ (each being measured by $\frac{1}{2}$ arc $AP$);

hence, § 302, $\triangle APF$ and $DBP$ are similar,
and, § 299, $PF : PD = AP : PB$;
similarly, $\triangle ADP$ and $PBE$ are similar,
and $PD : PE = AP : PB$.
Hence, Ax. 1, $PF : PD = PD : PE$.

Ex. **478**. From a point $A$ tangents $AB$ and $AC$ are drawn to a circle whose center is $O$, and $BD$ is drawn perpendicular to $CO$ produced. Prove that $BD$ is a fourth proportional to $AC$, $CD$, and $CO$.



**Proof.** Draw $BC$ and $OA$ intersecting as at $E$.

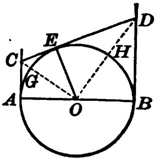
In $\triangle ABE$ and $ACE$, Ex. 221, $\angle BAE = \angle CAE$,
§ 231, $\angle ABE = \angle ACE$
(each being measured by $\frac{1}{2}$ arc $BC$);
$\therefore$ § 113, $\angle AEB = \angle AEC$; hence, § 26, $AO \perp BC$.

Then, in $\triangle ACO$ and $CDB$, since, § 205, $AC \perp CD$, and, data, $CO \perp BD$, § 307, the $\triangle$ are similar, and, § 299, $AC : CD = CO : BD$.

Ex. **479**. From a point $E$ in the common base of two triangles, $ABC$ and $ABD$, straight lines are drawn parallel to $AC$ and $AD$, meeting $BC$ and $BD$ at $F$ and $G$ respectively. Prove that $FG$ is parallel to $CD$.

**Proof.** § 301, $\triangle ABC$ and $EBF$ are similar;
$\therefore$ § 299, $AC : EF = AB : EB$;
similarly, from $\triangle ABD$ and $EBG$, $AD : EG = AB : EB$;
hence, $AC : EF = AD : EG$, and since, § 81, $\angle CAD = \angle FEG$,
§ 306, $\triangle ADC$ and $EGF$ are similar, and, § 299, $\angle ACD = \angle EFG$;
but, § 76, $\angle ACB = \angle EFB$; $\therefore$ $\angle BCD = \angle BFG$; hence, § 77, $FG \parallel CD$.

Ex. **480**. If tangents to a circle are drawn at the extremities of a diameter, the radius is a mean proportional between the segments of any third tangent intercepted between them and divided at its point of tangency.



**Data**: Tangents $AC$ and $BD$ at the extremities of diameter $AB$; a third tangent $CD$ meeting $AC$ and $BD$ in $C$ and $D$ respectively; and $OE$ the radius to the point of contact of $CD$.

**To prove** $CE : OE = OE : ED$.

**Proof.** Draw $CO$ and $DO$.

Then, Ex. 221, arc $GE = \frac{1}{2}$ arc $AE$, and arc $EH = \frac{1}{2}$ arc $EB$;
hence, arc $GEH = \frac{1}{2}$ arc $AEB = 90°$;
consequently, $\angle COD$ is a rt. $\angle$, and since, § 205, $OE \perp CD$,
§ 312, $CE : OE = OE : ED$.

## BOOK V

**Ex. 481.** Triangles on the same base and having their vertices in the same line which is parallel to the base are equivalent.

**Proof.** Since, § 151, parallel lines are everywhere equally distant, these triangles have equal altitudes; hence, § 336, they are equivalent.

Ex. **482.** The parallel sides of a trapezoid are $12^{\text{dm}}$ and $8^{\text{dm}}$, and their distance apart is $5^{\text{dm}}$. What is the area of the trapezoid?

**Solution.** § 338, $\text{area} = \frac{1}{2} \times 5(12 + 8) = 50^{\text{sq dm}}$.

Ex. **483.** The area of a trapezoid is 52 sq. in., and the sum of the two parallel sides is 13 in. What is the distance between the parallel sides?

**Solution.** Denote the altitude by $h$.

Then, § 338, $52 = \frac{13}{2} h$; whence, $h = 8$ ft.

Ex. **484.** The area of a triangle is 36 sq. ft. If its base is 9 ft., what is its altitude?

**Solution.** § 335, $36 = \frac{9}{2} h$; whence, $h = 8$ ft.

Ex. **485.** Two triangles have an angle in each equal, the including sides in one being 8 ft. and 12 ft., and in the other 6 ft. and 20 ft. The area of the smaller triangle is 27 sq. ft. Find the area of the larger triangle.

**Solution.** Let $x$ denote the number of sq. ft. in the larger $\triangle$.

Then, § 340, $27 : x = 8 \cdot 12 : 6 \cdot 20 = 96 : 120$;

whence, $$x = \frac{27 \times 120}{96} = 33\tfrac{3}{4} \text{ sq. ft.}$$

Ex. **486.** The homologous sides of two similar triangular fields are in the ratio of 5 : 3. How many times the area of the second field is the area of the first?

**Solution.** § 342, 1st field : 2d field $= 5^2 : 3^2 = 25 : 9$; consequently, the area of the 1st field is $2\frac{7}{9}$ times the area of the 2d field.

Ex. **487.** In two similar polygons two homologous sides are 15 ft. and 25 ft. The area of the smaller polygon is 450 sq. ft. Find the area of the larger one.

**Solution.** Let $x$ denote the number of square feet in the area of the larger polygon.

Then, § 344, $450 : x = 15^2 : 25^2$.

whence, $$x = \frac{450 \times 25^2}{15^2} = 1250 \text{ sq. ft.}$$

**Prop. XXII**, page 185.

Prove that $BDKL \Bumpeq BCGF$.

**Proof.** Draw $AF$ and $CD$.

In $\triangle$s $BDC$ and $ABF$, $BD = AB$, $BC = BF$,

and $\angle DBC = \angle ABF$; $\therefore$ § 100, $\triangle BDC = \triangle ABF$;

but, § 334, $BDKL \Bumpeq 2\triangle BDC$, and $BCGF \Bumpeq 2\triangle ABF$;

hence, $BDKL \Bumpeq BCGF$.

Ex. **488**. Prove $CE$ perpendicular to $JB$.

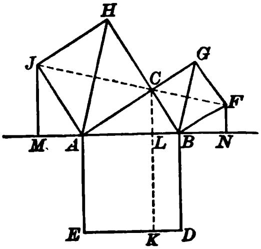
**Proof.** Let $M$ and $N$ be the points of intersection of $JB$ with $AC$ and $CE$ respectively. In proof of Prop. XII, it was shown that $\triangle AEC = \triangle AJB$;

$\because$ § 108, $\angle ACE = \angle AJB$, and, § 59, $\angle CMN = \angle AMJ$;

$\therefore$ § 113, $\angle MNC = \angle MAJ$.

But $\angle MAJ$ is a rt. $\angle$; $\therefore$ $\angle MNC$ is a rt. $\angle$; that is, $CE \perp JB$.

Ex. **489**. If the lines $AH$ and $BG$ are drawn, prove that they are parallel.



**Proof.** $AC = CH$, and $\triangle ACH$ is a rt. isos. $\triangle$;

consequently, $\angle CAH = 45°$;

similarly, $\angle CGB = 45°$;

$\therefore$ $\angle CAH = \angle CGB$, and, § 75, $AH \parallel BG$.

Ex. **490**. Prove that the sum of the perpendiculars from $F$ and $J$ to $AB$ produced is equal to $AB$.

**Proof.** (See Fig. for Ex. 489.) Data, $AJ \perp AC$, and $BC \perp AC$;

$\therefore$ § 71, $AJ \parallel BC$;

hence, § 76, $\angle MAJ = \angle CBL$; consequently, § 113, $\angle AJM = \angle CAL$,

and $AC = AJ$; $\therefore$ § 114, $\triangle AMJ = \triangle ALC$, and $JM = AL$.

Similarly, $FN = LB$; hence, Ax. 1, $JM + FN = AL + LB = AB$.

Ex. **491**. Prove that $J$, $C$, and $F$ are in the same straight line.

**Proof.** (See Fig. for Ex. 489.) Draw $CJ$ and $CF$.

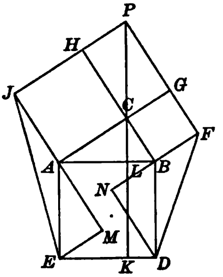
$CH = HJ$, $\triangle JCH$ is a rt. isos. $\triangle$, and $\angle JCH = \frac{1}{2}$ rt. $\angle$;

similarly, $\angle FCG = \frac{1}{2}$ rt. $\angle$, and since $\angle HCG$ is a rt. $\angle$,

$\angle JCH + \angle HCG + \angle FCG = 2$ rt. $\angle$s, or $\angle JCG + \angle FCG = 2$ rt. $\angle$s;

hence, § 58, $JCF$ is a straight line.

**Ex. 492.** If the lines $EJ$ and $DF$ are drawn, prove that the sum of the angles $AEJ$, $AJE$, $BDF$, and $BFD$ is equal to one right angle.



**Proof.** § 57, the sum of the $\angle$s about the point $B = 4$ rt. $\angle$s;

but, since $\angle$s $ABD$ and $CBF$ are rt. $\angle$s,

$$\angle ABC + \angle DBF = 2 \text{ rt. } \angle s.$$

Similarly, $\angle BAC + \angle EAJ = 2$ rt. $\angle$s;

$\therefore$ $\angle ABC + \angle BAC + \angle DBF + \angle EAJ = 4$ rt. $\angle$s;

but, § 111, $\angle ABC + \angle BAC = 1$ rt. $\angle$;

$\therefore$ Ax. 3, $\angle DBF + \angle EAJ = 3$ rt. $\angle$s.

But, § 110, $\angle AEJ + \angle AJE + \angle EAJ + \angle BDF$
$+ \angle BFD + \angle DBF = 4$ rt. $\angle$s;

hence, subtracting $\angle DBF + EAJ = 3$ rt. $\angle$s,

Ax. 3, $\angle AEJ + \angle AJE + \angle BDF + \angle BFD = 1$ rt. $\angle$.

Ex. **493.** If $EM$ and $DN$ are drawn perpendicular respectively to $JA$ and $FB$ produced, prove that the triangles $AEM$ and $BDN$ are each equal to triangle $ABC$.

**Proof.** (See Fig. for Ex. 492.) $AM \perp AC$, and $AE \perp AB$;

$\therefore$ § 83, $\angle EAM = \angle BAC$,

and since $AE = AB$, § 114, $\triangle AEM = \triangle ABC$.

Also, $BN \perp BC$, and $BD \perp AB$; $\therefore$ $\angle DBN = \angle ABC$,

and since $DB = AB$, $\triangle BDN = \triangle ABC$.

Ex. **494.** If the lines $JH$, $KC$, and $FG$ are produced, prove that they meet in a common point.

**Proof.** (See Fig. for Ex. 492.) Produce two of the lines, as $JH$ and $FG$, until they meet as at $P$, and draw $PC$.

§ 151, $HP = CG$, § 143, $BC = CG$; $\therefore$ $HP = BC$; also $HC = AC$,

and, § 52, $\angle CHP = \angle ACB$;

hence, § 100, $\triangle CPH = \triangle ABC$, and $\angle HCP = \angle BAC$;

but in rt. $\triangle ALC$, § 111, $\angle ACL + \angle LAC = 1$ rt. $\angle$;

$\therefore$ $\angle ACL + \angle HCP = 1$ rt. $\angle$,

and since $\angle ACH$ is a rt. $\angle$, $\angle ACL + \angle ACH + \angle HCP = 2$ rt. $\angle$s,

that is, Ax. 9, $\angle ACK + \angle ACP = 2$ rt. $\angle$s;

hence, § 58, $KCP$ is a straight line;

that is, $JH$, $KC$, and $FG$ produced meet in a common point, $P$.

Ex. **495.** If $D$ is the middle point of the side $BC$ of the right triangle $ABC$, and $DE$ is drawn perpendicular to the hypotenuse $AB$, prove that $\overline{AC}^2 = \overline{AE}^2 - \overline{BE}^2$.

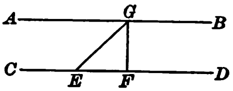
**Proof.** Draw $AD$. § 349, $\overline{AC}^2 \Bumpeq \overline{AD}^2 - \overline{DC}^2$; (1)
but $\overline{AD}^2 \Bumpeq \overline{AE}^2 + \overline{DE}^2$, and $\overline{DC}^2 = \overline{DB}^2 \Bumpeq \overline{BE}^2 + \overline{DE}^2$;
∴ substituting in (1), $\overline{AC}^2 \Bumpeq \overline{AE}^2 + \overline{DE}^2 - (\overline{BE}^2 + \overline{DE}^2)$;
that is, $\overline{AC}^2 \Bumpeq \overline{AE}^2 - \overline{BE}^2$.

Ex. **496**. The area of a rectangle is $26.40^{\text{sq dm}}$ and its altitude is $4.8^{\text{dm}}$. Find the length of its diagonal.

**Solution.** Let $b$ denote the number of decimeters in the base.

Then, § 330, $26.40 = b \times 4.8$; whence, $b = 5.5^{\text{dm}}$;
consequently, from § 349, the diagonal $= \sqrt{4.8^2 + 5.5^2} = 7.3^{\text{dm}}$.

Ex. **497**. The perpendicular distance between two parallel lines is 20 in. and a line is drawn across them at an angle of 45°. What is the length of the part intercepted between the parallel lines?



**Solution.** Let $AB$ be parallel to $CD$ and let the line drawn across them intersect them in $G$ and $E$ respectively, making $\angle GED = 45°$. Draw $GF \perp AB$.
Then, since

$$\angle GEF = \tfrac{1}{2} \text{ rt. } \angle, \ \S\ 111,\ \angle EGF = \tfrac{1}{2} \text{ rt. } \angle,$$

and, § 118, $EF = FG = 20$ in.
Hence, § 349, $\overline{EG}^2 = 20^2 + 20^2 = 800$, and $EG = \sqrt{800} = 28.284+$ in.

Ex. **498**. Find the area of a right isosceles triangle, if the hypotenuse is 100 rd. in length.

**Solution.** Let $x$ denote the number of rods in each of the equal sides.

Then, § 349, $100^2 = x^2 + x^2$; whence, $x^2 = \dfrac{100^2}{2}$;

but, § 335, $\text{area} = \tfrac{1}{2} x \cdot x = \tfrac{1}{2} x^2 = \dfrac{\overline{100}^2}{4} = 2500$ sq. rd.

Ex. **499**. The diameter of a circle is $12^{\text{cm}}$, and a chord of the circle is $10^{\text{cm}}$. What is the length of a perpendicular from the center to this chord?

**Solution.** Let $AB$ be the chord in the circle whose center is $O$, and $OD$ the given ⊥. Draw $AO$.

Then, in rt. $\triangle ADO$, § 200, $AD = \tfrac{1}{2} AB = 5$, and since $AO = 6$,
§ 350, $OD = \sqrt{6^2 - 5^2} = 3.316+^{\text{cm}}$.

Ex. **500**. Two parallel chords in a circle are each 8 ft. in length, and the distance between them is 6 ft. Find the radius of the circle.

**Solution.** Since the chords are equal, § 202, they are equally distant from the center; hence, the ⊥ from the center to one of the chords is 3 ft., and, § 200, it bisects the chord. Draw a radius to one extremity of this chord thus forming a rt. △ whose hypotenuse is the radius.

Then, the radius $= \sqrt{4^2 + 3^2} = 5$ ft.

**Ex. 501.** Two sides of a triangle are $13^{dm}$ and $15^{dm}$, and the altitude on the third side is $12^{dm}$. Find the third side, and also the area of the triangle.

**Solution.** Let $x$ and $y$ denote the number of decimeters in the segments of the third side adjacent respectively to the $13^{dm}$ and $15^{dm}$ sides.

Then, § 350, $x = \sqrt{13^2 - 12^2} = 5^{dm}$, and $y = \sqrt{15^2 - 12^2} = 9^{dm}$;

hence, the required side $= x + y = 5^{dm} + 9^{dm} = 14^{dm}$.

§ 335, area of the $\triangle = \frac{1}{2} \cdot 14 \cdot 12 = 84^{sq\,dm}$.

Ex. **502.** Two parallel lines are 12 ft. apart, and from a point on one of them two lines, one 20 ft. and the other 13 ft. long, are drawn to the other parallel. What is the area of the triangle thus formed?

**Solution.** Two sides of the $\triangle$ are 20 ft. and 13 ft. Let fall a $\perp$ upon the third side from the vertex opposite, dividing the side into the segments $x$ and $y$ respectively.

Then, § 350, $x = \sqrt{20^2 - 12^2} = 16$ ft., and $y = \sqrt{13^2 - 12^2} = 5$ ft.;

consequently, the third side $= x + y = 16$ ft. $+ 5$ ft. $= 21$ ft.,

and, § 335, the area $= \frac{1}{2} \times 21 \times 12 = 126$ sq. ft.

**Sch.** If the lines connecting the parallels are drawn on the *same* side of the perpendicular, the third side is 16 ft. – 5 ft. = 11 ft.; and the area of the triangle is $\frac{1}{2} \times 11 \times 12 = 66$ sq. ft.

**Ex. 503.** The diagonals of a rhombus are 30 in. and 16 in. What is the length of the sides?

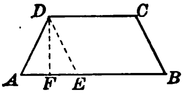
**Solution.** Since, Ex. 111, the diagonals of a rhombus bisect each other at rt. $\measuredangle$s, a half of each diagonal and a side of the rhombus will form a rt. $\triangle$ whose hypotenuse is a side of the rhombus;

hence, § 349, each side $= \sqrt{15^2 + 8^2} = 17$ in.

Ex. **504**. A square lawn with the walk around it contains $\frac{1}{10}$ of an acre. If the walk contains $\frac{15}{64}$ of the entire area, what is the width of the walk?

**Solution.** $\frac{1}{10}$ A. = 16 sq. rd.; $\therefore$ the side of lawn $= \sqrt{16} = 4$ rd. Area of walk $= \frac{15}{64}$ of 16 sq. rd. $= 3\frac{3}{4}$ sq. rd.; $\therefore$ area of lawn inside of the walk $= 16$ sq. rd. $- 3\frac{3}{4}$ sq. rd. $= 12\frac{1}{4}$ sq. rd., or $\frac{49}{4}$ sq. rd., and its side $= \sqrt{\frac{49}{4}} = \frac{7}{2}$, or $3\frac{1}{2}$ rd.; consequently, 4 rd. $- 3\frac{1}{2}$ rd. $= \frac{1}{2}$ rd., twice the width of the walk, and the width of the walk is $\frac{1}{4}$ rd., or 4 ft. $1\frac{1}{2}$ in.

**Ex. 505.** Find the area of a field in the form of a trapezoid whose bases are 45 rd. and 27 rd., and each of whose non-parallel sides is 15 rd.



**Solution.** Let $ABCD$ be the trapezoid with $AB = 45$ rd., $DC = 27$ rd., and $AD = BC = 15$ rd.

Draw $DE \parallel CB$ meeting $AB$ as in $E$, and draw $DF \perp AB$.

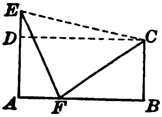
§ 76, $\angle AED = \angle B$, and, Ex. 138, $\angle A = \angle B$; $\therefore \angle A = \angle AED$; hence, § 118, $\triangle AED$ is isosceles, and, § 122,

$AF = FE = \frac{1}{2}(AB - EB) = \frac{1}{2}(AB - DC) = \frac{1}{2}$ (45 rd. − 27 rd.) = 9 rd.

§ 350, $DF = \sqrt{\overline{AD}^2 - \overline{AF}^2} = \sqrt{15^2 - 9^2} = \sqrt{144} = 12$ rd.

Then, § 338, area of $ABCD = \frac{1}{2}DF(AB + DC) = \frac{1}{2} \times 12(45 + 27) = 432$ sq. rd.

**Ex. 506.** A ladder $25^{\text{m}}$ long, with its foot in the street, will reach on one side to a window $20^{\text{m}}$ high, and on the other to a window $15^{\text{m}}$ high. What is the distance between the windows?



**Solution.**

Let $AE = 20^{\text{m}}$, $BC = 15^{\text{m}}$, and $FE = FC = 25^{\text{m}}$.

§ 350, $AF = \sqrt{\overline{FE}^2 - \overline{AE}^2} = \sqrt{25^2 - 20^2} = \sqrt{225} = 15^{\text{m}}$,

and $FB = \sqrt{\overline{FC}^2 - \overline{BC}^2} = \sqrt{25^2 - 15^2} = \sqrt{400} = 20^{\text{m}}$;

hence, $AB = AF + FB = 15^{\text{m}} + 20^{\text{m}} = 35^{\text{m}}$, the width of the street;

that is, $DC = 35^{\text{m}}$. Also $DE = AE - BC = 20^{\text{m}} - 15^{\text{m}} = 5^{\text{m}}$.

Hence, $EC = \sqrt{\overline{DC}^2 + \overline{DE}^2} = \sqrt{35^2 + 5^2} = \sqrt{1250} = 35.355+^{\text{m}}$, the distance between the windows.

**Ex. 507.** Find the area of a triangle each of whose sides is 12 ft.

**Solution.** Let fall a $\perp$ from the vertex to the base.

Then, § 122, this $\perp$ bisects the base and forms a rt. $\triangle$ with half the base, and one of the sides of the given $\triangle$ as hypotenuse.

Then, § 350, the $\perp = \sqrt{12^2 - 6^2} = 6\sqrt{3}$;

consequently, the area of the original $\triangle = \frac{1}{2} \cdot 12 \cdot 6\sqrt{3} = 62.352+$ sq. ft.

**Ex. 508.** The side of a rhombus is $29^{\text{cm}}$ and one of its diagonals is $40^{\text{cm}}$. What is the length of the other diagonal?

**Solution.** Ex. 111, the diagonals of a rhombus bisect each other at rt. $\angle$s; $\therefore$ a half of each diagonal and a side of the rhombus forms a rt. $\triangle$ whose hypotenuse is the side of the rhombus;

consequently, § 350, $\frac{1}{2}$ the required diagonal $= \sqrt{29^2 - 20^2} = 21^{\text{cm}}$,

and the required diagonal $= 42^{\text{cm}}$.

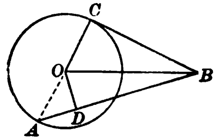
**Ex. 509.** The area of a rhombus is 1176 sq. in. and one of its diagonals is 42 in. What are its sides and the other diagonal?

**Solution.** Since, Ex. 111, the diagonals bisect each other at rt. $\angle$s, one half of the given diagonal is the altitude of each of the $\triangle$s into which the rhombus is divided by the required diagonal, which forms the common base of the $\triangle$s.

Then, § 335, area of rhombus $= \frac{1}{2}(21 \times \text{req. diag.}) + \frac{1}{2}(21 \times \text{req. diag.})$
$= 21 \times \text{req. diag.} = 1176$; whence, req. diag. $= \frac{1176}{21} = 56$ in.

§ 349, each side of the rhombus $= \sqrt{28^2 + 21^2} = 35$ in.

**Ex. 510.** The radius of a circle is $8^{\text{dm}}$ and a tangent to the circle is $15^{\text{dm}}$. What is the length of a secant drawn from the same point as the tangent, if the secant is $5^{\text{dm}}$ from the center?



**Solution.** Let the tangent $BC$ to the ⊙ whose center is $O$, be $15^{\text{dm}}$, the ⊥, $OD$ from $O$ to the secant $AB$, be $5^{\text{dm}}$, and let the radius of the ⊙ be $8^{\text{dm}}$.

Then, since, § 205, $\angle OCB$ is a rt. $\angle$,

§ 349, $OB = \sqrt{\overline{BC}^2 + \overline{OC}^2} = \sqrt{15^2 + 8^2} = \sqrt{289} = 17^{\text{dm}}$.

§ 350, $DB = \sqrt{\overline{OB}^2 - \overline{OD}^2} = \sqrt{17^2 - 5^2} = \sqrt{264} = 16.247+^{\text{dm}}$.

Also, $AD = \sqrt{\overline{OA}^2 - \overline{OD}^2} = \sqrt{8^2 - 5^2} = \sqrt{39} = 6.244+^{\text{dm}}$.

Hence, $AB = AD + DB = 6.244+^{\text{dm}} + 16.247+^{\text{dm}} = 22.491+^{\text{dm}}$.

**Ex. 511.** A pole standing on level ground was broken 75 ft. from the top and fell so that the end struck 60 ft. from the foot. Find the length of the pole.

**Solution.** § 350, the part left standing $= \sqrt{75^2 - 60^2} = 45$ ft.;
consequently, the length of the pole $= 45$ ft. $+ 75$ ft. $= 120$ ft.

**Ex. 512.** Upon the diagonal of a rectangle $28^{\text{m}}$ by $21^{\text{m}}$ a triangle equivalent to the rectangle is constructed. What is the altitude of the triangle?

**Solution.** § 349, diag. = base of $\triangle = \sqrt{28^2 + 21^2} = 35^{\text{m}}$.
§ 330, area of rect. $= 28 \times 21 = 588^{\text{sq m}}$;
consequently, § 335, area of $\triangle = \frac{1}{2}(35 \times \text{alt.}) = 588^{\text{sq m}}$;
whence, the alt. $= 33.6^{\text{m}}$.

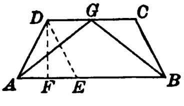
**Ex. 513.** The base and altitude of a right triangle are 6 ft. and 8 ft., respectively. What is the length of the perpendicular drawn from the vertex of the right angle to the hypotenuse?

**Solution.** Let $ABC$ be the given $\triangle$ and $BD$ the ⊥ to the hypotenuse $AC$.

Then, § 349, $AC = \sqrt{\overline{AB}^2 + \overline{BC}^2} = \sqrt{6^2 + 8^2} = 10$ ft.
Now, $\triangle$s $ABD$ and $ABC$ have $\angle A$ in common;
∴ § 302, the $\triangle$s are similar, and $AB : AC = BD : BC$;
that is, $6 : 10 = BD : 8$; whence, $BD = \dfrac{6 \times 8}{10} = 4.8$ ft.

**Ex. 514.** The parallel sides of a trapezoid are 12 in. and 16 in. and the non-parallel sides are 10 in. What is the area of the triangle formed by joining the middle point of the shorter base with the extremities of the longer?

**Solution.** Let $ABCD$ be the trapezoid with $AB =$ 16 in., $DC = 12$ in., $AD = BC = 10$ in., and let $G$ the middle point of $DC$ be joined to $A$ and $B$ by lines $AG$ and $BG$.



Proceeding as in Ex. 505, $AF = \frac{1}{2}(AB - DC) = \frac{1}{2}(16 - 12) = 2$ in.,
and the alt. $DF = \sqrt{\overline{AD}^2 - \overline{AF}^2} = \sqrt{96} = 9.7979+$ in.

Then, § 335, area $\triangle ABG = \frac{1}{2} AB \times DF = \frac{1}{2} \times 16 \times 9.7979+ = 78.38+$ sq. in.

**Prop.** XX, page 195.

**Proof.** § 349, $\overline{ED}^2 \Leftrightarrow \overline{FD}^2 + \overline{FE}^2$; that is, $C \Leftrightarrow A + B$.

**Prop.** XXI, page 195.

**Proof.** § 349, $\overline{EF}^2 \Leftrightarrow \overline{GF}^2 + \overline{GE}^2$; whence, $\overline{GF}^2 \Leftrightarrow \overline{EF}^2 - \overline{GE}^2$;
that is, $C \Leftrightarrow A - B$.

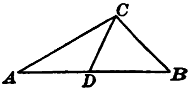
**Prop.** XXII, page 196.

**Proof.** § 349, $\overline{GE}^2 \Leftrightarrow \overline{FE}^2 + \overline{FG}^2$, and $\overline{FE}^2 \Leftrightarrow \overline{DE}^2 + \overline{DF}^2$;
hence, $\overline{GE}^2 \Leftrightarrow \overline{DE}^2 + \overline{DF}^2 + \overline{FG}^2$; that is, $H \Leftrightarrow A + B + C$.

**Ex. 515.** Divide a triangle into two equivalent triangles by a line drawn through any vertex.

**Solution.** From any vertex, as $C$, of the given $\triangle ABC$, draw a line $CD$ to bisect the opposite side, $AB$.

Then, § 336, $CD$ divides $\triangle ABC$ into two equivalent $\triangle$s.

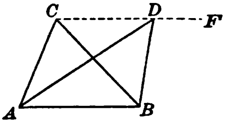


**Ex. 516.** Construct a triangle equivalent to a given triangle and having the same base.

**Solution.** Through $C$, the vertex of the given $\triangle ABC$, draw $CF \parallel AB$; select any point in $CF$, as $D$; draw $AD$ and $BD$.

Then, Ex. 481, $\triangle ABD$ is the $\triangle$ required.

**Sch.** Since $\angle BAD$ may be of different sizes, there may be an indefinite number of $\triangle$s fulfilling the conditions.



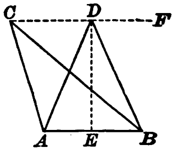
**Ex. 517.** Construct an isosceles triangle equivalent to a given triangle and having the same base.

**Solution.** Through $C$, the vertex of the given $\triangle ABC$, draw $CF \parallel AB$.

At $E$, the middle point of $AB$, erect a $\perp$ and produce it to intersect $CF$, as at $D$. Draw $AD$ and $BD$.

Then, $\triangle ABD$ is the $\triangle$ required.

**Proof.** § 103, $AD = BD$; $\therefore \triangle ABD$ is isosceles, and, Ex. 481, $\triangle ABD \Leftrightarrow \triangle ABC$.



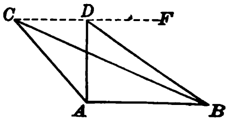
**Ex. 518.** Construct a right triangle equivalent to a given triangle.

**Solution.** Through the vertex $C$ of the given $\triangle ABC$ draw $CF \parallel AB$.

At either extremity of $AB$, as $A$, erect a $\perp$ to $AB$ and produce it to meet $CF$, as in $D$. Draw $BD$.

Then, $\triangle ABD$ is the $\triangle$ required.

**Proof.** Const., $AD \perp AB$; $\therefore \triangle ABD$ is a rt. $\triangle$, and, Ex. 481, $\triangle ABD \Leftrightarrow \triangle ABC$.



**Sch.** Other rt. $\triangle$s equivalent to $\triangle ABC$ may be constructed by taking the base and altitude of such length that their product shall be equal to the product of $AB$ and $AD$; *e.g.* a rt. $\triangle$ whose base is $2\,AB$ and altitude $\frac{1}{2}AD$ would be equivalent to $\triangle ABC$.

Ex. **519.** Construct a triangle equivalent to a given triangle, having the same base and an angle at the base equal to a given angle.

**Solution.** (See Fig. for Ex. 516.) Through $C$ the vertex of the given $\triangle ABC$, draw $CF \parallel AB$.

Construct $\angle BAD =$ the given $\angle$, the side $AD$ meeting $CF$ as at $D$. Draw $BD$.

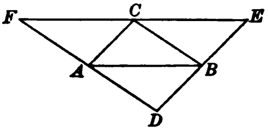
Then, Ex. 481, $\triangle ABD$ is the required $\triangle$.

Ex. **520.** Construct a triangle similar to a given triangle and four times the given triangle.

**Solution.** Through the vertices $A$, $B$, and $C$ of the given $\triangle$ draw lines parallel to the opposite sides respectively, and produce them to intersect as in $D$, $E$, and $F$.

Then, $\triangle DEF$ is the $\triangle$ required.

**Proof.** § 307, $\triangle$s $DEF$ and $ABC$ are similar, and, Ex. 164, $\triangle DEF \Leftrightarrow 4\,\triangle ABC$.

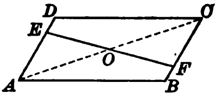


Ex. **521.** Divide a parallelogram into two equivalent parts by a line through any point in its perimeter.

**Solution.** Draw the diagonal $AC$ of the given ▱ $ABCD$ and find its middle point $O$.

From $E$ any given point in the perimeter draw $EO$ and produce it to meet the perimeter again as at $F$.

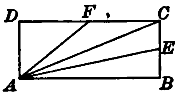
Then, Ex. 173, $EF$ divides $ABCD$ into two equivalent parts.



Ex. **522.** Divide a rectangle into four equivalent parts by lines through any vertex.

**Solution.** From any vertex as $A$ of the given rectangle $ABCD$ draw diagonal $AC$ and the lines $AE$ and $AF$ to the middle points of $BC$ and $DC$ respectively.

Then, $ABCD$ is divided into four equivalent parts.



**Proof.** § 336, $\triangle ABE \Leftrightarrow \triangle AEC$, and $\triangle ACF \Leftrightarrow \triangle AFD$;
but, § 152, $\triangle ABC = \triangle ACD$; $\therefore$ $\triangle ABE \Leftrightarrow \triangle AEC \Leftrightarrow \triangle ACF \Leftrightarrow \triangle AFD$.

Ex. **523.** Construct a square equivalent to a triangle whose base is $18^{\text{cm}}$ and altitude $4^{\text{cm}}$.

**Solution.** § 335, area of $\triangle = \frac{1}{2}(4 \times 18) = 36^{\text{sq cm}}$;

$\therefore$ the side of the required square is $\sqrt{36} = 6^{\text{cm}}$.

Then, Ex. 288, construct a square whose side is $6^{\text{cm}}$.

Ex. **524**. Construct a square equivalent to a rectangle whose dimensions are $16^{cm}$ and $4^{cm}$.

**Solution.** § 330, area of the rectangle $= 16 \times 4 = 64^{sq\ cm}$;

$\therefore$ the side of the required square is $\sqrt{64} = 8^{cm}$.

Then, Ex. 288, construct a square whose side is $8^{cm}$.

Ex. **525**. Construct a square equivalent to the difference between two squares whose areas are $25^{sq\ cm}$ and $16^{sq\ cm}$.

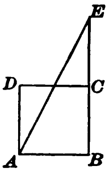
**Solution.** The area of the required square $= 25^{sq\ cm} - 16^{sq\ cm} = 9^{sq\ cm}$;

$\therefore$ its side $= \sqrt{9} = 3^{cm}$.

Then, Ex. 288, construct a square whose side is $3^{cm}$.

Ex. **526**. Construct a right triangle equivalent to a given square.

**Solution.** Produce the side $BC$ of the given square $ABCD$ to $E$ making $CE = BC$. Draw $AE$.



Then, $\triangle ABE$ is the $\triangle$ required.

**Proof.** § 330, area $ABCD = AB \times BC$,

and, § 335, area $\triangle ABE = \frac{1}{2} AB \times BE = \frac{1}{2} AB \times 2BC = AB \times BC$;

hence, rt. $\triangle ABE \Leftrightarrow ABCD$.

Ex. **527**. Construct a right triangle equivalent to a given rectangle.

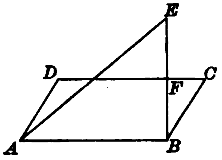
**Solution.** Produce the side $BC$ of the given rectangle $ABCD$ to $E$ making $CE = BC$. Draw $AE$.

Then, $\triangle ABE$ is the $\triangle$ required.

**Proof.** The same as for Ex. 526.

Ex. **528**. Construct a right triangle equivalent to a given parallelogram.

**Solution.** Draw the altitude $BF$ of the given parallelogram $ABCD$ and produce it to $E$, making $FE = BF$. Draw $AE$.



Then, $\triangle ABE$ is the $\triangle$ required.

**Proof.** § 332, area $ABCD = AB \times BF$,

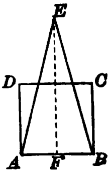
and, § 335, area rt. $\triangle ABE =$

$\frac{1}{2} AB \times BE = \frac{1}{2} AB \times 2BF = AB \times BF$;

hence, $\triangle ABE \Leftrightarrow ABCD$.

Ex. **529**. Construct an isosceles triangle equivalent to a given square.

**Solution.** At $F$, the middle point of the base $AB$ of the given square $ABCD$, erect the perpendicular $FE = 2BC$. Draw $AE$ and $BE$.



Then, $\triangle ABE$ is the $\triangle$ required.

**Proof.** Since, § 103, $AE = BE$, $\triangle ABE$ is isosceles.

As in Ex. 526, $\triangle ABE \Leftrightarrow ABCD$.

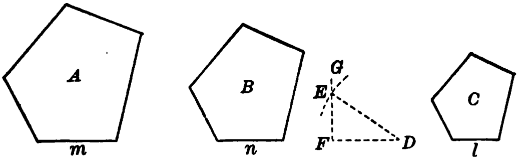
**Ex. 530.** Construct a square equivalent to the sum of two squares whose sides are 5 in. and 10 in.

**Solution.** Proceed as in § 362, having a side of $A = 10$ in., and a side of $B = 5$ in.

**Ex. 531.** Construct a square equivalent to the difference of two squares whose sides are $15^{\text{cm}}$ and $17^{\text{cm}}$.

**Solution.** Proceed as in § 363, having a side of $A = 17^{\text{cm}}$ and a side of $B = 15^{\text{cm}}$.

Ex. **532.** Construct a polygon similar to two given similar polygons, and equivalent to their difference.



**Solution.** Draw a line as $FD = n$, a side of $B$ the smaller of the given polygons.

At $F$ erect a perpendicular $FG$ and from $D$ as a center with a radius $= m$, the side of $A$, the other given polygon, homologous to $n$, describe an arc cutting $FG$ as at $E$.

Take $l = FE$, and on $l$ homologous to $m$ and $n$, construct polygon $C$, similar to $A$ and $B$.

Then, $C$ is the required polygon.

**Proof.** $\overline{FE}^2 = \overline{DE}^2 - \overline{FD}^2$; $\therefore\ l^2 = m^2 - n^2$.

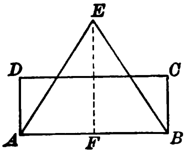
§ 344, $A : C = m^2 : l^2$, or $\dfrac{A}{C} = \dfrac{m^2}{l^2}$,

and $B : C = n^2 : l^2$, or $\dfrac{B}{C} = \dfrac{n^2}{l^2}$;

$\therefore$ subtracting, $\dfrac{A - B}{C} = \dfrac{m^2 - n^2}{l^2} = 1.$

Hence, $C \approx A - B$.

Ex. **533.** Construct an isosceles triangle equivalent to a given rectangle.

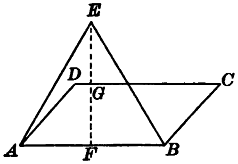


**Solution.** At $F$, the middle point of the base $AB$ of the given rectangle $ABCD$, erect the perpendicular $FE = 2\,BC$. Draw $AE$ and $BE$.

Then, $\triangle ABE$ is the $\triangle$ required.

**Proof.** Proceed as in Ex 529.

**Ex. 534.** Construct an isosceles triangle equivalent to a given parallelogram.

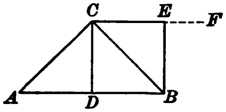


**Solution.** At $F$, the middle point of the base $AB$ of the given parallelogram $ABCD$, erect the perpendicular $FE = 2\,FG$, the altitude of $ABCD$. Draw $AE$ and $BE$.

Then, $\triangle ABE$ is the $\triangle$ required.

**Proof.** Since, § 103, $AE = BE$, $\triangle ABE$ is isosceles, and as in Ex. 528, $\triangle ABE \Bumpeq ABCD$.

**Ex. 535.** Construct a square which shall be equivalent to a right isosceles triangle, having given the perpendicular from the vertex of the right angle upon the hypotenuse.



**Solution.** Through $C$, the vertex of the rt. $\angle$ of the given $\triangle ABC$, draw $CF \parallel AB$; draw $BE \parallel DC$, the given altitude, meeting $CF$ as in $E$.

Then, $DBEC$ is the required square.

**Proof.** § 140, $DBEC$ is a ▱, and since $\angle CDB$ is a rt. $\angle$, Ex. 104, $DBEC$ is a rectangle.

Since, § 122, $D$ is the middle point of $AB$, Ex. 163, $DC = \frac{1}{2} AB = DB$; hence, § 153, all the sides of $DBEC$ are equal and it is a square.

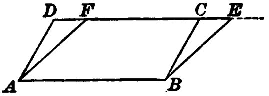
§ 335, area $\triangle ABC = \frac{1}{2} AB \times DC$,

and, § 330, area $DBEC = DB \times DC = \frac{1}{2} AB \times DC$;

hence, $DBEC \Bumpeq \triangle ABC$.

**Ex. 536.** Construct a parallelogram equivalent to a given parallelogram and having an angle equal to a given angle.

**Solution.** At one extremity, as $A$, of the base $AB$ of the given ▱ $ABCD$, construct $\angle BAF =$ the given $\angle$, the side $AF$ meeting $DC$, or $DC$ produced, as in $F$.



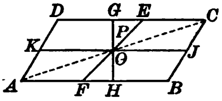
Draw $BE \parallel AF$ meeting $DC$, or $DC$ produced, as in $E$.

Then, $ABEF$ is the required parallelogram.

**Proof.** Since, § 151, parallel lines are everywhere equally distant, $ABEF$ and $ABCD$ have equal altitudes;

hence, § 333, $ABEF \Bumpeq ABCD$.

**Ex. 537.** Bisect a given parallelogram (1) by a line passing through a given point within; (2) by a line perpendicular to a side; (3) by a line parallel to a side.



**Solution.** (1) Draw the diagonal $AC$ of the given ▱ $ABCD$ and find its middle point $O$.

Through $O$ and $P$, the given point within, draw a line terminating in the perimeter as in $E$ and $F$.

Then, Ex. 173, $EF$ bisects $ABCD$.

(2) Through $O$ draw a line perpendicular to any side, as $AB$, terminating in the perimeter as in $H$ and $G$.

Then, Ex. 173, $HG$ bisects $ABCD$.

(3) Through $O$ draw a line parallel to any side, as $AB$, terminating in the perimeter, as in $K$ and $J$.

Then, Ex. 173, $KJ$ bisects $ABCD$.

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**Proof.** § 314, $FG : JG = JG : GK$.

$\therefore$ § 269, $\overline{JG}^2 = GK \times FG = AB \times BE$; that is, $H \Leftrightarrow ABCD$.

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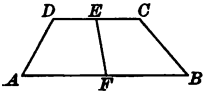
**Proof.** § 315, $DF : BD = BD : DE$;

$\therefore$ § 269, $DF \times DE = \overline{BD}^2$, that is, $H \Leftrightarrow A$.

Ex. **538**. The straight line joining the middle points of the parallel sides of a trapezoid bisects the trapezoid.

**Data:** Trapezoid $ABCD$, and $EF$ joining the middle points of its parallel sides.



**To prove** $EF$ bisects $ABCD$.

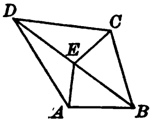
**Proof.** Let $h$ denote the altitude of $ABCD$.

§ 338, area of $ABCD = \frac{1}{2}h(AB + DC)$,

and area of $AFED = \frac{1}{2}h(\frac{1}{2}AB + \frac{1}{2}DC) = \frac{1}{4}h(AB + DC)$;

consequently, $AFED \Leftrightarrow \frac{1}{2}ABCD$; that is, $EF$ bisects $ABCD$.

Ex. **539**. The lines joining the middle point of either diagonal of a quadrilateral to the opposite vertices divide the quadrilateral into two equivalent parts.



**Data:** Quadrilateral $ABCD$ and either diagonal as $BD$; also lines $EA$ and $EC$ drawn from $E$ the middle point of $BD$ to the vertices $A$ and $C$ respectively.

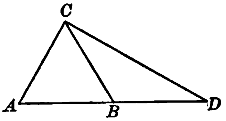
**To prove** $ABCE \Leftrightarrow AECD$.

**Proof.** § 336, $\triangle ABE \Leftrightarrow \triangle AED$, and $\triangle BCE \Leftrightarrow \triangle DEC$;

hence, $\triangle ABE + \triangle BCE \Leftrightarrow \triangle AED + \triangle DEC$;

that is, $ABCE \Leftrightarrow AECD$.

Ex. **540**. Two triangles are equivalent, if they have two sides of one respectively equal to two sides of the other, and if the included angles are supplementary.



**Data:** $\triangle$s $ABC$ and $BDC$ having $AB = BD$, $BC$ in common, and $\angle ABC$ the supplement of $\angle DBC$.

**To prove** $\triangle ABC \Leftrightarrow \triangle BDC$.

**Proof.** Since, data, $\angle ABC + \angle DBC = 2$ rt. $\angle$s, § 58, $ABD$ is a straight line, and the altitude of $\triangle ABC$ equals the altitude of $\triangle BDC$, and since, data, $AB = BD$,

§ 336, $\triangle ABC \Leftrightarrow \triangle BDC$.

Ex. **541.** $O$ is any point on the diagonal $AC$ of the parallelogram $ABCD$. If the lines $DO$ and $BO$ are drawn, prove that the triangles $AOB$ and $AOD$ are equivalent.

**Proof.** Since, § 152, $\triangle ABC = \triangle ADC$, their altitudes on the side $AC$ as a base must be equal; consequently, the altitudes of $\triangle$s $AOB$ and $AOD$ on $AO$ as a base are equal.

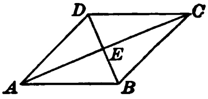
Hence, § 336, $\triangle AOB \Bumpeq \triangle AOD$.

Ex. **542.** A rhombus and a rectangle have equal bases and equal areas. One side of the rhombus is $15^m$ and the altitude of the rectangle is $12^m$. What are their perimeters?

**Solution.** Since, § 144, the sides of the rhombus are equal, its perimeter $= 15^m \times 4 = 60^m$.

Since the base of the rectangle is $15^m$ and its altitude or perpendicular side is $12^m$, its perimeter $= 2(15^m + 12^m) = 54^m$.

Ex. **543.** The area of a rhombus is equal to one half the product of its diagonals.



**Data**: Rhombus $ABCD$, and its diagonals $AC$ and $BD$ intersecting at $E$.

**To prove** area $ABCD = \frac{1}{2} AC \times BD$.

**Proof.** Since, Ex. 111, the diagonals of a rhombus are at rt. $\angle$s,

§ 335, area $\triangle ABC = \frac{1}{2} AC \times BE$, and area $\triangle ADC = \frac{1}{2} AC \times ED$; consequently, area $ABCD = \frac{1}{2} AC \times (BE + ED) = \frac{1}{2} AC \times BD$.

Ex. **544.** The diagonals of a rhombus are 64 rd. and 37 rd. What is the area of the rhombus?

**Solution.** Ex. 543, area of rhombus $= \frac{1}{2}(64 \times 37) = 1184$ sq. rd.

Ex. **545.** The base of a triangle is $75^m$, and its altitude is $60^m$. Find the perimeter of an equivalent rhombus, if its altitude is $45^m$.

**Solution.** Denote the base of the rhombus by $b$.

§ 335, area of the $\triangle = \frac{1}{2}(75 \times 60) = 2250^{sq\,m}$;

hence, § 332, data, area of rhombus $= b \times 45 = 2250^{sq\,m}$;

whence, $b = 50^m$, and perimeter of rhombus $= 50^m \times 4 = 200^m$.

Ex. **546.** Find the area of a rhombus, if the sum of its diagonals is 12 in. and their ratio 3 : 5.

**Solution.** Let $x =$ the longer diagonal; then, $12 - x =$ the shorter one.

Then, $12 - x : x = 3 : 5$; whence, $x = 7\frac{1}{2}$ in., and $12 - x = 4\frac{1}{2}$ in.

Hence, Ex. 543, area of the rhombus $= \frac{1}{2}(7\frac{1}{2} \times 4\frac{1}{2}) = 16\frac{7}{8}$ sq. in.

**Ex. 547.** A man travels 25 miles east from a certain town, and another man travels 36 miles north from the same town. How far apart are the men?

**Solution.** § 349, the required distance $=\sqrt{25^2+36^2}=43.82+$ miles.

**Ex. 548.** The shortest side of a triangle acute-angled at the base is 45 ft. long, and the segments of the base made by a perpendicular from the vertex are 27 ft. and 77 ft. How long is the other side?

**Solution.** § 350, the perpendicular $=\sqrt{45^2-27^2}$;

$\therefore$ § 349,

the required side $=\sqrt{(\sqrt{45^2-27^2})^2+77^2}=\sqrt{45^2+77^2-27^2}=85$ ft.

**Ex. 549.** The sides of a triangle are $25^{\text{m}}$ and $17^{\text{m}}$, and the lesser segment of the base made by a perpendicular from the vertex is $8^{\text{m}}$. What is the length of the base?

**Solution.** § 350, the perpendicular $=\sqrt{17^2-8^2}$;

$\therefore$ § 350,

the greater segment of the base $=\sqrt{25^2-(\sqrt{17^2-8^2})^2}=\sqrt{25^2-17^2+8^2}=20^{\text{m}}$;

consequently, the base $=20^{\text{m}}+8^{\text{m}}=28^{\text{m}}$.

**Ex. 550.** In a right triangle the base is $3^{\text{dm}}$, and the difference between the hypotenuse and perpendicular is $1^{\text{dm}}$. What are the hypotenuse and perpendicular?

**Solution.** Let $x$ denote the perpendicular; then, data, the hypotenuse $=1+x$.

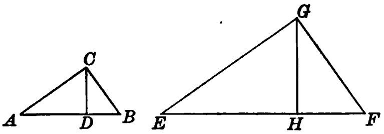
Hence, § 349, $3^2+x^2=(1+x)^2$, or $9+x^2=1+2x+x^2$; whence, $x=4^{\text{dm}}$, the perpendicular, and $1+x=5^{\text{dm}}$, the hypotenuse.

**Ex. 551.** In a right triangle the hypotenuse is $5^{\text{dm}}$, and the difference between the base and perpendicular is $1^{\text{dm}}$. Find the base and perpendicular.

**Solution.** Let $x$ denote the base; then, data, the perpendicular $=x+1$.

Hence, $5^2=x^2+(x+1)^2$; whence, $x^2+x=12$, and $x=3^{\text{dm}}$, the base; consequently, the perpendicular $=x+1=4^{\text{dm}}$.

**Ex. 552.** The sides of a right triangle are in the ratio of 3, 4, and 5, and the perpendicular upon the hypotenuse from the vertex of the right angle is 20 yd. What is the area of the triangle?



**Solution.** Let $ABC$ be a rt. $\Delta$ whose sides $BC$, $AC$, and $AB$ are 3 yd., 4 yd., and 5 yd. respectively, and whose altitude is $CD$, and let $EFG$ be the rt. $\Delta$ whose sides are in the ratio of 3, 4, and 5, the $\perp CH$ upon the hypotenuse being 20 yd.

§ 302, $\triangle$s $ABC$ and $BDC$ are similar, and $AB : BC = AC : CD$;

that is, $5 : 3 = 4 : CD$; whence, $CD = \frac{12}{5}$.

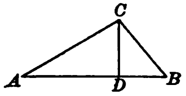
§§ 303, 305, $GH : CD = EF : AB$; that is, $20 : \frac{12}{5} = EF : 5$;

whence, $EF = \dfrac{20 \times 5 \times 5}{12} = \dfrac{125}{3}$ yd.

Hence, § 335, area of $\triangle EFG = \frac{1}{2} EF \times GH = \frac{1}{2} \times \frac{125}{3} \times 20 = 416\frac{2}{3}$ sq. yd.

**Ex. 553.** If in any triangle a perpendicular is drawn from the vertex to the base, the difference of the squares upon the sides is equivalent to the difference of the squares upon the segments of the base.

**Data**: $\triangle ABC$, in which $AC > BC$, also the $\perp CD$ from the vertex $C$ to the base $AB$.



**To prove** $\overline{AC}^2 - \overline{BC}^2 \Leftrightarrow \overline{AD}^2 - \overline{BD}^2$.

**Proof.** § 349, $\overline{AC}^2 \Leftrightarrow \overline{AD}^2 + \overline{CD}^2$,

and $\overline{BC}^2 \Leftrightarrow \overline{BD}^2 + \overline{CD}^2$;

hence, subtracting, $\overline{AC}^2 - \overline{BC}^2 \Leftrightarrow \overline{AD}^2 - \overline{BD}^2$.

**Ex. 554.** In a right triangle the square on either side containing the right angle is equivalent to the rectangle contained by the sum and the difference of the other two sides.

**Data**: Rt. $\triangle ABC$ whose rt. $\angle$ is at $C$, and either side about the rt. $\angle$, as $AC$.

**To prove** $\overline{AC}^2 \Leftrightarrow$ rect. $(AB + BC) \cdot (AB - BC)$.

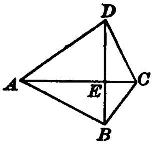
**Proof.** § 350, $AC = \sqrt{\overline{AB}^2 - \overline{BC}^2}$, or $\overline{AC}^2 = \overline{AB}^2 - \overline{BC}^2$;

hence, $\overline{AC}^2 = (AB + BC)(AB - BC)$;

that is, § 324, $\overline{AC}^2 \Leftrightarrow$ rect. $(AB + BC) \cdot (AB - BC)$.

**Ex. 555.** If the diagonals of a quadrilateral intersect at right angles, prove that the sum of the squares upon one pair of opposite sides is equivalent to the sum of the squares upon the other pair.

**Data**: Quadrilateral $ABCD$ whose diagonals $AC$ and $BD$ intersect at rt. $\angle$s in $E$.



**To prove** $\overline{AB}^2 + \overline{DC}^2 \Leftrightarrow \overline{AD}^2 + \overline{BC}^2$.

**Proof.** § 349, $\overline{AB}^2 \Leftrightarrow \overline{AE}^2 + \overline{BE}^2$, and $\overline{DC}^2 \Leftrightarrow \overline{DE}^2 + \overline{CE}^2$;

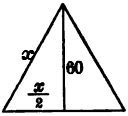
$\therefore$ $\overline{AB}^2 + \overline{DC}^2 \Leftrightarrow \overline{AE}^2 + \overline{BE}^2 + \overline{DE}^2 + \overline{CE}^2$.

Similarly, $\overline{AD}^2 + \overline{BC}^2 \Leftrightarrow \overline{AE}^2 + \overline{DE}^2 + \overline{BE}^2 + \overline{CE}^2$.

Hence, $\overline{AB}^2 + \overline{DC}^2 \Leftrightarrow \overline{AD}^2 + \overline{BC}^2$.

Ex. **556**. The altitude of an equilateral triangle is 60 in. How long are its sides?

**Solution.** Let $x$ denote the side of the $\triangle$; then, § 122, each segment of the base made by the perpendicular from the vertex $= \frac{1}{2}x$.



§ 349, $x^2 = 60^2 + (\frac{1}{2}x)^2$, or $x^2 - \frac{1}{4}x^2 = 3600$; whence, $x^2 = 4800$, and $x = 69.28+$ in.

Ex. **557**. Through $D$ and $E$, the middle points of the sides $AC$ and $BC$ of the triangle $ABC$, any two parallel straight lines are drawn meeting $AB$ or $AB$ produced in the points $F$ and $G$. Prove that the parallelogram $DFGE$ is equivalent to half the triangle $ABC$.

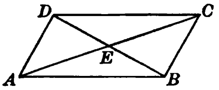
**Proof.** Denote the altitude of the $\triangle$ by $h$; then, the altitude of $\square = \frac{1}{2}h$.

§ 335, area $\triangle ABC = \frac{1}{2}AB \times h$, and, § 332, area $DFGE = FG \times \frac{1}{2}h$;

but, §§ 153, 158, $FG = DE = \frac{1}{2}AB$; $\therefore$ area $DFGE = \frac{1}{4}AB \times h$;

hence, $$DFGE \Leftrightarrow \tfrac{1}{2}\triangle ABC.$$

Ex. **558**. The four triangles into which a parallelogram is divided by its diagonals are equivalent.



**Data**: $\square ABCD$ and its diagonals intersecting at $E$.

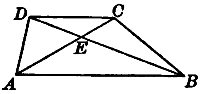
**To prove** $\triangle AED \Leftrightarrow \triangle DEC \Leftrightarrow \triangle BEC \Leftrightarrow \triangle ABE$.

**Proof.** § 154, $AE = CE$, and $DE = BE$;

hence, § 336, $\triangle AED \Leftrightarrow \triangle DEC$, $\triangle BEC \Leftrightarrow \triangle DEC$, and $\triangle BEC \Leftrightarrow \triangle ABE$;

consequently, $$\triangle AED \Leftrightarrow \triangle DEC \Leftrightarrow \triangle BEC \Leftrightarrow \triangle ABE.$$

Ex. **559**. The diagonals of a trapezoid divide it into four triangles, two of which are similar, while the other two are equivalent.



**Data**: Trapezoid $ABCD$ divided by its diagonals into the $\triangle$s $ABE$, $DCE$, $AED$, and $BEC$.

**To prove** two of these $\triangle$s similar and the other two equivalent.

**Proof.** § 73, $\angle EAB = \angle ECD$, and $\angle ABE = \angle EDC$;

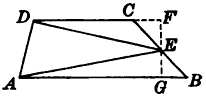
hence, § 301, $\triangle$s $ABE$ and $DCE$ are similar.

Also, § 336, $$\triangle ADC \Leftrightarrow \triangle BDC;$$

$\therefore$ subtracting $\triangle DEC$ from each,

$$\triangle AED \Leftrightarrow \triangle BEC.$$

Ex. **560**. In any trapezoid the triangle having for its base one of the non-parallel sides and for its vertex the middle point of the opposite side, is equivalent to one half of the trapezoid.



**Data**: Trapezoid $ABCD$ and the $\triangle AED$ formed by $AD$ and the lines $AE$ and $DE$ drawn from $A$ and $D$ to $E$, the middle point of $BC$.

**To prove** $\triangle AED \Leftrightarrow \frac{1}{2} ABCD$.

**Proof.** Through $E$ draw $GF \perp AB$ meeting $DC$ produced in $F$.

Then, data, $BE = EC$, and, § 59, $\angle BEG = \angle CEF$;

∴ § 114, $\triangle BGE = \triangle CFE$, and $GE = EF = \frac{1}{2} GF$.

§ 334, $\triangle ABE \Leftrightarrow \frac{1}{2}$ rect. $AB \cdot \frac{1}{2} GF$, and $\triangle DCE = \frac{1}{2}$ rect. $DC \cdot \frac{1}{2} GF$;

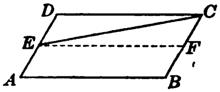
hence, $\triangle ABE + \triangle DCE \Leftrightarrow \frac{1}{4}$ rect. $(AB + DC) \cdot GF$;

but, § 337, $ABCD \Leftrightarrow \frac{1}{2}$ rect. $(AB + DC) \cdot GF$;

∴ $\triangle ABE + \triangle DCE \Leftrightarrow \frac{1}{2} ABCD$;

consequently, $ABCD - (\triangle ABE + \triangle DCE) \Leftrightarrow \triangle AED \Leftrightarrow \frac{1}{2} ABCD$.

Ex. **561.** The triangle, formed by drawing a line from any vertex of a parallelogram to the middle point of one of the opposite sides, is equivalent to one fourth of the parallelogram.



**Data**: ▱ $ABCD$ and $\triangle ECD$ cut off by $CE$ drawn from the vertex $C$ to the middle point of $AD$.

**To prove** $\triangle ECD \Leftrightarrow \frac{1}{4} ABCD$.

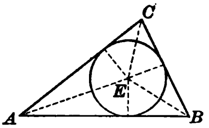
**Proof.** Draw $EF \parallel AB$ meeting $BC$ in $F$.

Since, § 157, $EF$ bisects the altitude of $ABCD$,

§ 333, $ABFE \Leftrightarrow EFCD$; but, § 152, $\triangle ECD = \frac{1}{2} EFCD$;

hence, $\triangle ECD \Leftrightarrow \frac{1}{4} ABCD$.

Ex. **562.** A triangle is equivalent to one half the rectangle of its perimeter and the radius of the inscribed circle.



**Data**: $\triangle ABC$ and the inscribed circle whose radius is $r$ and center $E$.

**To prove** $\triangle ABC \Leftrightarrow \frac{1}{2}$ rect. $(AB + BC + AC) \cdot r$.

**Proof.** Draw $AE$, $BE$, and $CE$ and the radii to the points of contact.

Then, since, § 205, these radii are perpendicular to the sides respectively,

§ 334, $\triangle ABE \Leftrightarrow \frac{1}{2}$ rect. $AB \cdot r$,

$\triangle BEC \Leftrightarrow \frac{1}{2}$ rect. $BC \cdot r$,

and $\triangle AEC \Leftrightarrow \frac{1}{2}$ rect. $AC \cdot r$;

hence, $\triangle ABE + \triangle BEC + \triangle AEC \Leftrightarrow \frac{1}{2}$ rect. $(AB + BC + AC) \cdot r$;

that is, $\triangle ABC \Leftrightarrow \frac{1}{2}$ rect. $(AB + BC + AC) \cdot r$.

Ex. **563**. If the perimeter of a quadrilateral circumscribed about a circle is 400 ft. and the radius of the circle is 25 ft., what is the area of the quadrilateral?

**Solution.** Area of quadrilateral $= \frac{1}{2} \times 400 \times 25 = 5000$ sq. ft.

Ex. **564**. The area of a triangle is 875 sq. yd. Find its base and altitude, if they are in the ratio of 14 to 5.

**Solution.** § 335, the area of a similar $\triangle$ whose base is 14 yd. and altitude 5 yd. $= \frac{1}{2}(14 \times 5) = 35$ sq. yd.

Then, denoting the base of the given $\triangle$ by $b$ and its altitude by $h$,

§ 342, $b^2 : 14^2 = 875 : 35$, whence, $b = \sqrt{4900} = 70$ yd.

Also, § 343, $h^2 : 5^2 = 875 : 35$, whence, $h = \sqrt{625} = 25$ yd.

Ex. **565**. The homologous sides of two similar fields are in the ratio of 3 to 5, and the sum of their areas is $416^{\text{Ha}}$. What is the area of each field?

**Solution.** § 344, 1st field : 2d field $= 3^2 : 5^2$;

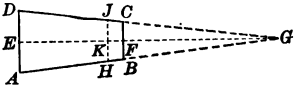
§ 276, 1st field + 2d field : 1st field $= 3^2 + 5^2 : 3^2$;

that is, 416 : 1st field $= 34 : 9$;

whence, area 1st field $= \frac{416 \times 9}{34} = 110.1176+^{\text{Ha}}$;

consequently, area 2d field $= 416^{\text{Ha}} - 110.1176+^{\text{Ha}} = 305.8824-^{\text{Ha}}$.

Ex. **566**. A board 12 ft. long is 10 in. wide at one end and 6 in. wide at the other. What length must be cut from the narrower end to contain a square foot?



**Solution.** Suppose the edges $AB$ and $DC$ of the board $ABCD$ be produced to meet as at $G$; then, §§ 301, 305, $EG : FG = AD : BC$;

that is, $12 + FG : FG = \frac{5}{6} : \frac{1}{2}$;

whence, $\frac{5}{6} FG = \frac{1}{2}(12 + FG)$, or $FG = 18$ ft.

Let $KF$ denote the length to be cut off.

Then, §§ 301, 305, $\triangle HGJ : \triangle BGC = \overline{GK}^2 : \overline{GF}^2$;

but area $\triangle BGC = \frac{1}{2} BC \times FG = \frac{1}{2} \times \frac{1}{2} \times 18 = 4\frac{1}{2}$ sq. ft.;

hence, area $\triangle HGJ = 4\frac{1}{2}$ sq. ft. $+ 1$ sq. ft. $= 5\frac{1}{2}$ sq. ft.;

consequently, $5\frac{1}{2} : 4\frac{1}{2} = (KF + 18)^2 : 18^2$; whence, $KF = 1.9$ ft.

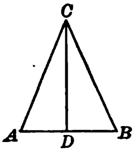
Ex. **567**. The side of one equilateral triangle is equal to the altitude of another. What is the ratio of their areas?

**Solution.** Denote the side of one by $a$, then, its altitude

$$= \sqrt{a^2 - \frac{a^2}{4}} = \sqrt{\frac{3a^2}{4}} = \text{the side of the other } \triangle;$$

hence, § 342, smaller $\triangle$ : larger $\triangle = \frac{3a^2}{4} : a^2 = 3a^2 : 4a^2 = 3 : 4$.

**Ex. 568.** The perimeter of an isosceles triangle whose base is its shortest side is $100^{\text{dm}}$; the difference between the base and an adjacent side is $23^{\text{dm}}$. What is the altitude of the triangle? What is its area?



**Solution.** In $\triangle ABC$ let $AB + BC + AC = 100^{\text{dm}}$

and $AC - AB = 23^{\text{dm}}$

subtracting, $2\,AB + BC = 77^{\text{dm}}$

Now, $BC - AB = 23^{\text{dm}}$

whence, subtracting, $3\,AB = 54^{\text{dm}}$

and $AB = 18^{\text{dm}}$; hence, if $CD$ is the $\perp$ from the vertex, $AD = 9^{\text{dm}}$. One of the equal sides $= 23^{\text{dm}} + 18^{\text{dm}} = 41^{\text{dm}}$;

then, $CD = \sqrt{41^2 - 9^2} = 40^{\text{dm}}$.

Hence, area $= \frac{1}{2} AB \times CD = \frac{1}{2} \cdot 18 \times 40 = 360^{\text{sq dm}}$.

**Ex. 569.** Two chords on opposite sides of the center of a circle are parallel; one is 16 ft. long, and the other is 12 ft. If the distance between them is 14 ft., what is the diameter of the circle?

**Solution.** Draw a line through the center $\perp$ to the chords; draw radii to the extremities of the chords, and denote the radius by $r$.

Then, § 350, $\sqrt{r^2 - 64} + \sqrt{r^2 - 36} = 14$.

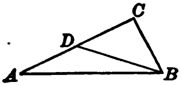
Squaring, $r^2 - 64 + 2\sqrt{r^2 - 64}\sqrt{r^2 - 36} + r^2 - 36 = 196$;

whence, $\sqrt{r^4 - 100\,r^2 + 2304} = 148 - r^2$;

squaring, $r^4 - 100\,r^2 + 2304 = 21{,}904 - 296\,r^2 + r^4$;

whence, $196\,r^2 = 19{,}600$, $r^2 = 100$, and $r = 10$ ft.

**Ex. 570.** If from the vertex of an acute angle of a right triangle a straight line is drawn, bisecting the opposite side, the square upon that line is less than the square upon the hypotenuse by three times the square upon half the line bisected.



**Data:** Rt. $\triangle ABC$ right-angled at $C$, and $BD$ drawn from vertex $B$ to bisect $AC$, as at $D$.

**To prove** $\overline{AB}^2 - \overline{DB}^2 = 3\,\overline{DC}^2$.

**Proof.** § 349, $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$;

but $\overline{AC}^2 = 4\,\overline{DC}^2$;

$\therefore$ $\overline{AB}^2 = 4\,\overline{DC}^2 + \overline{BC}^2$.

Now, $\overline{DB}^2 = \overline{BC}^2 + \overline{DC}^2$;

hence, subtracting, $\overline{AB}^2 - \overline{DB}^2 = 4\,\overline{DC}^2 + \overline{BC}^2 - (\overline{BC}^2 + \overline{DC}^2)$;

that is, $\overline{AB}^2 - \overline{DB}^2 = 3\,\overline{DC}^2$.

Ex. **571**. In the right triangle $ABC$, $\overline{BC}^2 = 3\,\overline{AC}^2$. If $CD$ is drawn from the vertex of the right angle to the middle point of $AB$, angle $ACD$ equals 60°.

**Proof.** $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 = 3\,\overline{AC}^2 + \overline{AC}^2 = 4\,\overline{AC}^2$;

$\therefore$ $AB = 2\,AC$, and $\frac{1}{2}AB$, or $AD = AC$;

also, Ex. 163, $CD = AD$;

consequently, $\triangle ADC$ is equilateral;

hence, Ex. 74, $\angle ACD = 60°$.

Ex. **572**. If $ACB$ and $ADB$ are two angles inscribed in a semicircle, and $AE$ and $BF$ are drawn perpendicular to $CD$ produced, prove that

$$\overline{CE}^2 + \overline{CF}^2 = \overline{DE}^2 + \overline{DF}^2.$$

**Proof.** § 349, $\overline{CE}^2 = \overline{AC}^2 - \overline{AE}^2$, and $\overline{CF}^2 = \overline{BC}^2 - \overline{BF}^2$;

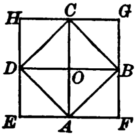
$\therefore$ $\overline{CE}^2 + \overline{CF}^2 = \overline{AC}^2 + \overline{BC}^2 - \overline{AE}^2 - \overline{BF}^2 = \overline{AB}^2 - (\overline{AE}^2 + \overline{BF}^2)$.

$\overline{DE}^2 = \overline{AD}^2 - \overline{AE}^2$, and $\overline{DF}^2 = \overline{BD}^2 - \overline{BF}^2$;

$\therefore$ $\overline{DE}^2 + \overline{DF}^2 = \overline{AD}^2 + \overline{BD}^2 - \overline{AE}^2 - \overline{BF}^2 = \overline{AB}^2 - (\overline{AE}^2 + \overline{BF}^2)$.

Hence, Ax. 1, $\overline{CE}^2 + \overline{CF}^2 = \overline{DE}^2 + \overline{DF}^2$.

Ex. **573**. If lines are drawn perpendicular to the diagonals of a square at their extremities, a second square is formed equivalent to twice the original square.



**Data:** Square $ABCD$ and lines $\perp$ to its diagonals through their extremities intersecting to form $EFGH$.

**To prove** $EFGH$ a square ≎ 2 $ABCD$.

**Proof.** Ex. 110, $\angle AOB$ is a rt. $\angle$, and, data, $\triangle$s $OAF$ and $OBF$ are rt. $\triangle$s;

$\angle F$ is a rt. $\angle$; similarly, $\angle$s $G$, $H$, and $E$ are rt. $\angle$s.

§ 71, $FG \parallel EH \parallel AC$, and $FE \parallel GH \parallel BD$;

$\therefore$ § 151, $FG = EH = AC$, and $FE = GH = BD$;

but, Ex. 112, $AC = BD$;

$\therefore$ $FG = EH = FE = GH$.

Hence, $EFGH$ is a square.

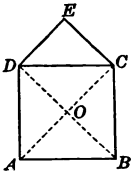
Now, $\overline{BD}^2 = \overline{AB}^2 + \overline{AD}^2 = 2\,\overline{AB}^2$;

but $BD = FE$; $\therefore$ $\overline{FE}^2 = 2\,\overline{AB}^2$;

consequently, $EFGH$ is a square ≎ 2 $ABCD$.

**Ex. 574.** The square upon the base of an isosceles triangle whose vertical angle is a right angle is equivalent to four times the triangle.

**Data :** Isos. $\triangle DCE$ and the square on $DC$, as $ABCD$.



**To prove** $ABCD \bumpeq 4 \triangle DCE.$

**Proof.** Draw the diagonals $AC$ and $BD$ intersecting as at $O$.

Since, Ex. 112, $AC = BD,$

§ 154, Ax. 7, $AO = BO = CO = DO,$

and, § 143, $AB = BC = DC = AD;$

hence, § 107, the diagonals divide $ABCD$ into 4 equal isos. $\triangle$s.

Data, $\angle E$ is a rt. $\angle$; $\therefore$ $\angle$s $EDC$ and $DCE$ each equal $\frac{1}{2}$ rt. $\angle$,

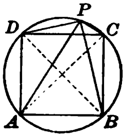
Ex. 110, $\angle DOC$ is a rt. $\angle$; $\therefore$ $\angle$s $ODC$ and $OCD$ each equal $\frac{1}{2}$ rt. $\angle$;

hence, § 102, $\triangle DCO = \triangle DCE;$

consequently, $ABCD = 4 \triangle DCE.$

**Ex. 575.** The sum of the squares on the lines joining any point in the circumference of a circle with the vertices of an inscribed square is equivalent to twice the square of the diameter of the circle.

**Data :** Inscribed square $ABCD$ and the lines $PA$, $PB$, $PC$, and $PD$ from any point $P$ in the circumference.



**To prove** $\overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2 + \overline{PD}^2 =$ twice the square of the diameter.

**Proof.** Draw the diagonals $AC$ and $BD$.

Since $AC$ and $BD$ are also diameters of the $\odot$,

§ 227, $\angle$s $APC$ and $BPD$ are rt. $\angle$s;

$\therefore$ § 349, $\overline{PA}^2 + \overline{PC}^2 = \overline{AC}^2,$

and $\overline{PD}^2 + \overline{PB}^2 = \overline{DB}^2;$

adding, $\overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2 + \overline{PD}^2 = \overline{AC}^2 + \overline{DB}^2 = 2\,\overline{AC}^2.$

**Ex. 576.** If $AF$ and $BE$ are the medians drawn from the extremities of the hypotenuse of the right $\triangle ABC$, prove that $4\,\overline{AF}^2 + 4\,\overline{BE}^2 = 5\,\overline{AB}^2$

**Proof.** § 349, $\overline{AF}^2 = \overline{AC}^2 + \overline{CF}^2$, and $\overline{BE}^2 = \overline{BC}^2 + \overline{CE}^2;$

$\therefore$ $\overline{AF}^2 + \overline{BE}^2 = \overline{AC}^2 + \overline{BC}^2 + \overline{CF}^2 + \overline{CE}^2.$

Draw $EF$; then, $\overline{CF}^2 + \overline{CE}^2 = \overline{EF}^2;$

and since, §§ 159, 158, $EF = \frac{1}{2} AB$, $\overline{EF}^2 = (\frac{1}{2} AB)^2 = \frac{1}{4} \overline{AB}^2;$

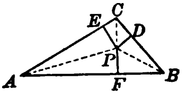
and since, $\overline{AC}^2 + \overline{BC}^2 = \overline{AB}^2,$

$\overline{AF}^2 + \overline{BE}^2 = \overline{AB}^2 + \frac{1}{4} \overline{AB}^2;$

consequently, $4\,\overline{AF}^2 + 4\,\overline{BE}^2 = 4\,\overline{AB}^2 + \overline{AB}^2 = 5\,\overline{AB}^2.$

**Ex. 577.** If perpendiculars $PF$, $PD$, and $PE$ are drawn from any point $P$ to the sides $AB$, $BC$, and $AC$ of a triangle, prove that

$$\overline{AF}^2 + \overline{BD}^2 + \overline{CE}^2 = \overline{AE}^2 + \overline{BF}^2 + \overline{CD}^2.$$



**Proof.** Draw $PA$, $PB$, and $PC$.

Then, § 349, $\overline{AF}^2 = \overline{AP}^2 - \overline{PF}^2$, $\overline{BD}^2 = \overline{BP}^2 - \overline{PD}^2$, and $\overline{CE}^2 = \overline{CP}^2 - \overline{PE}^2$;

$\therefore$ Ax. 2, $\overline{AF}^2 + \overline{BD}^2 + \overline{CE}^2 = \overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 - \overline{PF}^2 - \overline{PD}^2 - \overline{PE}^2$. (1)

Again, $\overline{AE}^2 = \overline{AP}^2 - \overline{PE}^2$, $\overline{BF}^2 = \overline{BP}^2 - \overline{PF}^2$, and $\overline{CD}^2 = \overline{CP}^2 - \overline{PD}^2$;

$\therefore$ Ax. 2, $\overline{AE}^2 + \overline{BF}^2 + \overline{CD}^2 = \overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 - \overline{PF}^2 - \overline{PD}^2 - \overline{PE}^2$. (2)

Hence, from (1) and (2), Ax. 1, $\overline{AF}^2 + \overline{BD}^2 + \overline{CE}^2 = \overline{AE}^2 + \overline{BF}^2 + \overline{CD}^2$.

**Ex. 578.** If any point $P$ within the rectangle $ABCD$ is joined to the vertices, prove that

$$\overline{PA}^2 + \overline{PC}^2 = \overline{PB}^2 + \overline{PD}^2.$$

**Proof.** Draw the $\perp$s $PF$, $PG$, $PE$, and $PH$ from $P$ to the sides $AB$, $BC$, $CD$, and $AD$ respectively.

Then, § 349, $\overline{PA}^2 = \overline{PF}^2 + \overline{AF}^2$, and $\overline{PC}^2 = (\overline{PE}^2 + \overline{EC}^2) = \overline{PE}^2 + \overline{FB}^2$;

$\therefore$ Ax. 2, $\overline{PA}^2 + \overline{PC}^2 = \overline{PF}^2 + \overline{PE}^2 + \overline{AF}^2 + \overline{FB}^2$. (1)

Also, $\overline{PB}^2 = \overline{PF}^2 + \overline{FB}^2$, and $\overline{PD}^2 = (\overline{PE}^2 + \overline{DE}^2) = \overline{PE}^2 + \overline{AF}^2$;

$\therefore$ $\overline{PB}^2 + \overline{PD}^2 = \overline{PF}^2 + \overline{PE}^2 + \overline{AF}^2 + \overline{FB}^2$. (2)

Hence, from (1) and (2), Ax. 1, $\overline{PA}^2 + \overline{PC}^2 = \overline{PB}^2 + \overline{PD}^2$.

**Ex. 579.** If $CD$ and $AE$ are the perpendiculars from the vertices $C$ and $A$ of the acute triangle $ABC$ to the opposite sides, prove that

$$\overline{AC}^2 = BC \times CE + AB \times AD.$$

**Proof.** § 353, $\overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2 - 2\,AB \times AD$,

and $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 - 2\,BC \times CE$.

Adding and transposing, $2\,\overline{AC}^2 = 2\,BC \times CE + 2\,AB \times AD$;

consequently, $\overline{AC}^2 = BC \times CE + AB \times AD$.

**Ex. 580.** If $AC$ and $BC$ are the equal sides of an isosceles triangle, and $AD$ is drawn perpendicular to $BC$, prove that

$$\overline{AB}^2 = 2\,BC \times BD.$$

**Proof.** § 353, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2\,BC \times BD$;

but, data, $AC = BC$; $\therefore$ $\overline{BC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2\,BC \times BD$;

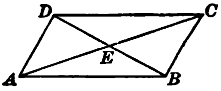
whence, $\overline{AB}^2 = 2\,BC \times BD$.

**Ex. 581.** The sum of the squares on the diagonals of a parallelogram is equivalent to the sum of the squares on its four sides.

**Data:** $\square ABCD$ whose diagonals are $AC$ and $BD$, intersecting at $E$.

**To prove** $\overline{AC}^2 + \overline{BD}^2 \Bumpeq \overline{AB}^2 + \overline{BC}^2 + \overline{DC}^2 + \overline{AD}^2$.

**Proof.** § 355, $\overline{AB}^2 + \overline{BC}^2 \Bumpeq 2\,\overline{AE}^2 + 2\,\overline{BE}^2$,



**and** $\overline{DC}^2 + \overline{AD}^2 \Bumpeq 2\,\overline{AE}^2 + 2\,\overline{DE}^2$;

but since, § 154, $DE = BE$, adding,

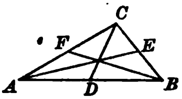
$$\overline{AB}^2 + \overline{BC}^2 + \overline{DC}^2 + \overline{AD}^2 \Bumpeq 4\,\overline{AE}^2 + 4\,\overline{BE}^2.$$

Now $AC = 2\,AE$; $\therefore\ \overline{AC}^2 \Bumpeq 4\,\overline{AE}^2$; similarly, $\overline{BD}^2 \Bumpeq 4\,\overline{BE}^2$,

and $\overline{AC}^2 + \overline{BD}^2 \Bumpeq 4\,\overline{AE}^2 + 4\,\overline{BE}^2$.

Hence, $\overline{AC}^2 + \overline{BD}^2 \Bumpeq \overline{AB}^2 + \overline{BC}^2 + \overline{DC}^2 + \overline{AD}^2$.

Ex. **582.** Three times the sum of the squares on the sides of a triangle is equivalent to four times the sum of the squares on the medians of the triangle.



**Data:** $\triangle ABC$ and its medians $AE$, $BF$, and $CD$.

**To prove**

$$3(\overline{AB}^2 + \overline{BC}^2 + \overline{AC}^2) \Bumpeq 4(\overline{AE}^2 + \overline{BF}^2 + \overline{CD}^2).$$

**Proof.** **§ 355,** $\overline{AB}^2 + \overline{AC}^2 \Bumpeq 2\left(\frac{BC}{2}\right)^2 + 2\,\overline{AE}^2$;

**that is,** $2\,\overline{AB}^2 + 2\,\overline{AC}^2 \Bumpeq \overline{BC}^2 + 4\,\overline{AE}^2$.

**Similarly,** $2\,\overline{BC}^2 + 2\,\overline{AB}^2 \Bumpeq \overline{AC}^2 + 4\,\overline{BF}^2$,

**and** $2\,\overline{AC}^2 + 2\,\overline{BC}^2 \Bumpeq \overline{AB}^2 + 4\,\overline{CD}^2$.

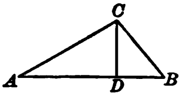
**Adding,**

$$4(\overline{AB}^2 + \overline{BC}^2 + \overline{AC}^2) \Bumpeq \overline{AB}^2 + \overline{BC}^2 + \overline{AC}^2 + 4(\overline{AE}^2 + \overline{BF}^2 + \overline{CD}^2);$$

hence, $3(\overline{AB}^2 + \overline{BC}^2 + \overline{AC}^2) \Bumpeq 4(\overline{AE}^2 + \overline{BF}^2 + \overline{CD}^2)$.

Ex. **583.** Two sides of an oblique triangle are 137 and 111 respectively, and the difference of the segments of the third side made by a perpendicular from the opposite vertex is 52. What is the third side?

**Solution.** In $\triangle ABC$, let $AC = 137$, $BC = 111$, and let $CD \perp AB$ divide $AB$, so that $AD - BD = 52$.



**§ 349,** $\overline{AD}^2 = 137^2 - \overline{CD}^2$,

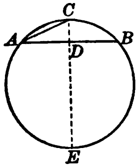
and $\overline{BD}^2 = 111^2 - \overline{CD}^2$;

$\therefore$ **subtracting,** $\overline{AD}^2 - \overline{BD}^2 = 137^2 - 111^2 = 6448.$ **(1)**

But, data, $AD - BD = 52.$ **(2)**

Dividing (1) by (2), $AD + BD = 6448 \div 52 = 124$, or $AB = 124$.

**Ex. 584.** The chord of an arc is 80 in.; the chord of half the arc is 41 in. What is the diameter of the circle?



**Solution.** Let chord $AB = 80$ in., and $AC$, the chord of half the arc $AB$, $= 41$ in.

Draw the diameter $CE$ from $C$, cutting $AB$ as at $D$.

Then, Ex. 188, $CE \perp AB$, and $D$ is the middle point of $AB$.

§ 350, $CD = \sqrt{\overline{AC}^2 - \overline{AD}^2} = \sqrt{41^2 - 40^2} = 9$ in.

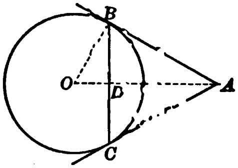
and, § 314, $DE : AD = AD : CD$;

that is, $DE : 40 = 40 : 9$;

whence, $DE = \frac{40 \times 40}{9} = 177\frac{7}{9}$ in.

Then, $CE = CD + DE = 9 \text{ in.} + 177\frac{7}{9} \text{ in.} = 186\frac{7}{9}$ in.

**Ex. 585.** From a point without a circle two tangents are drawn, which, with the chord of contact, form an equilateral triangle whose side is 18 in. Find the diameter of the circle.



**Solution.** Let $AB$ and $AC$ be tangent to the circle whose center is $O$, forming with the chord $BC$ an equilateral △.

Draw radius $OB$, and draw $AO$, cutting $BC$ as at $D$.

Then, § 106, $AO \perp BC$ at its middle point.

§ 312, $OD : BD = BD : AD$; that is, $OD : 9 = 9 : AD$;

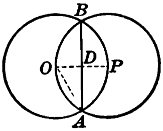
whence, $OD = \frac{81}{AD}$; but $AD = \sqrt{18^2 - 9^2} = \sqrt{243}$;

hence, $OD = \frac{81}{\sqrt{243}}$.

Then, $\text{diameter} = 2\,OB = 2\sqrt{\overline{OD}^2 + \overline{BD}^2} = 2\sqrt{\left(\frac{81}{\sqrt{243}}\right)^2 + 9^2}$

$= 2\sqrt{108} = 12\sqrt{3} = 20.784+$ in.

**Ex. 586.** If the center of each of two equal circles is on the circumference of the other, the square on the common chord is equivalent to three times the square on the radius.



**Data:** Equal circles of radius $r$ whose centers $O$ and $P$ are each on the circumference of the other. Also the common chord $AB$.

**To prove** $\overline{AB}^2 \eqcirc 3\,r^2$.

**Proof.** Draw the line of centers $OP$, cutting $AB$, as at $D$, and draw $AO$.

§ 212, $OP \perp AB$ at its middle point,

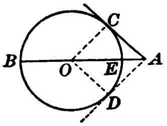
and since $A$ and $B$ are each equidistant from $O$ and $P$, $D$ is also the middle point of $OP$.

Then, $OD = \frac{1}{2}OP = \frac{1}{2}r$, $AD = \frac{1}{2}AB$, and $AO = r$.

Consequently, $(\frac{1}{2}AB)^2 \Leftrightarrow r^2 - (\frac{1}{2}r)^2$;

whence, $\overline{AB}^2 \Leftrightarrow 4r^2 - r^2 \Leftrightarrow 3r^2$.

Ex. **587**. A tangent and a secant meet without a circle, forming an angle of 45°; the tangent is 2 ft. long and the diameter of the circle is 4 ft. Find the length of the secant.



**Solution.** Let $AC = 2$ ft. be a tangent to the ⊙ whose center is $O$, and let $AB$ be a secant making $\angle CAB = 45°$, and cutting the arc as at $E$.

Draw $AD$ the other tangent from $A$, and draw $OC$, $OA$, and $OD$.

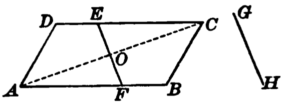
Data and § 209, each side of $ADOC$ equals 2 ft.; ∴ § 148, $ADOC$ is a ▱, and since, § 205, $\angle ACO$ is a rt. ∠, Ex. 104, all the ∠s of $ADOC$ are rt. ∠s; ∴ $ADOC$ is a square, and diagonal $AO$ bisects $\angle CAD$; that is, $\angle CAO = 45°$; but, data, $\angle CAB = 45°$; ∴ $AO$ and $AB$ take the same direction, $AB$ passes through $O$, and, data, $BE = 4$ ft.

Then, § 315, $AB : AC = AC : AE$; that is, $AE + 4 : 2 = 2 : AE$,

and, § 269, $\overline{AE}^2 + 4AE = 4$; whence, $AE = .828426+$ ft., or 9.94+ in.;

consequently, $AB = BE + AE = 4$ ft. 9.94 in.

Ex. **588**. Divide a given parallelogram into two equivalent parts by a line drawn parallel to a given line.

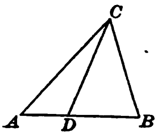


**Solution.** Draw the diagonal $AC$ of the given ▱ $ABCD$ and find its middle point $O$.

Through $O$ draw a line parallel to the given line $GH$ and terminating in the perimeter, as in $E$ and $F$.

Then, Ex. 173, $EF$ divides $ABCD$ into two equivalent parts.

Ex. **589**. Divide a given triangle into two parts, whose ratio is 3 : 4, by a line drawn from one vertex.



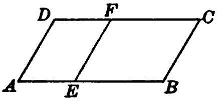
**Solution.** Divide any side, as $AB$, of the given $\triangle ABC$ into 7 equal parts; take $AD = 3$ of them.

From the vertex $C$ opposite the side $AB$ draw $CD$.

Then, $CD$ divides $\triangle ABC$ into the required parts.

**Proof.** § 336, $\triangle ADC : \triangle DBC = AD : DB = 3 : 4$.

**Ex. 590.** Divide a given parallelogram into two parts, whose ratio is 2 : 3, by a line parallel to a side.



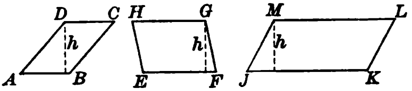
**Solution.** Divide any side, as $AB$, of the given ▱ $ABCD$ into 5 equal parts and take $AE = 2$ of them.

Draw $EF \parallel AD$ meeting $DC$ as in $F$.

Then, $EF$ divides $ABCD$ into the required parts.

**Proof.** § 333, $AEFD : EBCF = AE : EB = 2 : 3.$

Ex. **591**. Construct a parallelogram equivalent to the sum of two given parallelograms of equal altitude.



**Solution.** Denote the altitude of the given ▱s by $h$.

With base $JK = AB + EF$, the sum of the bases of the given ▱s $ABCD$ and $EFGH$, construct the ▱ $JKLM$, having its altitude $= h$.

Then, $JKLM$ is the required ▱.

**Proof.** § 331, $ABCD \Bumpeq$ rect. $AB \cdot h$, and $EFGH \Bumpeq$ rect. $EF \cdot h$;

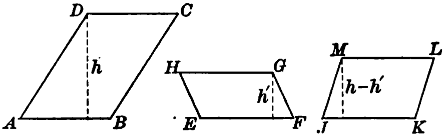
hence, $ABCD + EFGH \Bumpeq$ rect. $AB \cdot h +$ rect. $EF \cdot h \Bumpeq$ rect. $(AB + EF) \cdot h$.

But $JKLM \Bumpeq$ rect. $JK \cdot h \Bumpeq$ rect. $(AB + EF) \cdot h$.

Hence, $JKLM \Bumpeq ABCD + EFGH.$

**Sch.** Since the size of $\angle J$ is not fixed there is an unlimited number of solutions.

Ex. **592**. Construct a parallelogram equivalent to the difference of two given parallelograms of equal bases.



**Solution.** Draw the altitudes $h$ and $h'$ of the given ▱s $ABCD$ and $EFGH$, respectively.

On $JK = AB$ as base construct ▱ $JKLM$ with altitude $= h - h'$.

Then, $JKLM$ is the ▱ required.

**Proof.** § 331, $ABCD \Bumpeq$ rect. $AB \cdot h$,

and $EFGH \Bumpeq$ rect. $EF \cdot h' \Bumpeq$ rect. $AB \cdot h'$;

hence, $ABCD - EFGH \Bumpeq$ rect. $AB \cdot h -$ rect. $AB \cdot h' \Bumpeq$ rect. $AB \cdot (h - h')$.

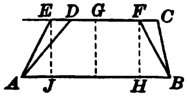
But $JKLM \Bumpeq$ rect. $JK \cdot (h - h') \Bumpeq$ rect. $AB \cdot (h - h')$

Hence, $JKLM \Bumpeq ABCD - EFGH.$

**Ex. 593.** Construct a square equivalent to five times a given square.

**Solution.** By the method employed in § 364, construct a square equivalent to the sum of five squares each equal to the given square.

**Ex. 594.** Transform a given trapezoid into an equivalent isosceles trapezoid.



**Solution.** Erect a perpendicular at the middle point of $AB$, the longer base of the given trapezoid $ABCD$, and produce it to cut $DC$, as at $G$.

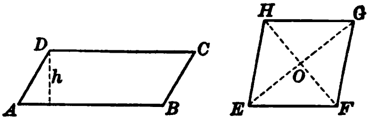
From $G$, on $DC$ or $DC$ produced, lay off $GF$ and $GE$ each equal to half $DC$. Draw $AE$ and $BF$.

Then, $ABFE$ is the trapezoid required.

**Proof.** $ABFE \Leftrightarrow ABCD$ (having equal bases and the same altitude).

Draw $EJ$ and $FH \perp AB$ and prove $ABFE$ isosceles, as in Ex. 324.

**Ex. 595.** Construct a rhombus equivalent to a given parallelogram, and having one side of the parallelogram for a diagonal.



**Solution.** Draw the altitude $h$ of the given $\square ABCD$ and let $AB$ be the given side.

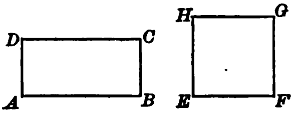
Draw $EG = AB$ and at its middle point $O$ draw $HOF \perp EG$, making $OH = OF = h$; draw $EF$, $FG$, $GH$, and $HE$.

Then, $EFGH$ is the required rhombus.

**Proof.** Ex. 543, area $EFGH = \frac{1}{2} EG \times HF = \frac{1}{2} AB \times 2h = AB \times h$; but, § 332, area $ABCD = AB \times h$; hence, $EFGH \Leftrightarrow ABCD$.

**Sch.** If $2h = AB$ the resulting figure is a square.

**Ex. 596.** Construct a square equivalent to a given rectangle.



**Solution.** § 319, find $EF$, the mean proportional between $AB$ and $AD$, the base and altitude respectively of the given rectangle $ABCD$.

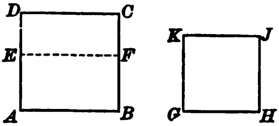
On $EF$ as a side construct the square $EFGH$.

Then, $EFGH$ is the required square.

**Proof.** $AB : EF = EF : AD$; $\therefore$ § 269, $\overline{EF}^2 = AB \times AD$;

that is, $EFGH \Leftrightarrow ABCD$.

**Ex. 597.** Construct a square equivalent to four sevenths of a given square.



**Solution.** On $AD$, a side of the given square $ABCD$, take $AE = \frac{4}{7} AD$, and draw $EF \parallel AB$, meeting $BC$ in $F$.

As in Ex. 596, construct square $GHJK$ equivalent to $ABFE$.

Then, $GHJK$ is the required square.

**Proof.** § 328, $ABFE : ABCD = AE : AD = 4 : 7$;

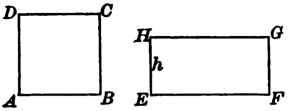
that is, $ABFE \Leftrightarrow \frac{4}{7} ABCD$; but, const., $GHJK \Leftrightarrow ABFE$:

hence, $GHJK \Leftrightarrow \frac{4}{7} ABCD$.

**Ex. 598.** Construct a rectangle equivalent to a given square and having a given side.

**Solution.** § 318, find $h$ a third proportional to $EF$, the given side, and $AB$ a side of the given square $ABCD$.

Construct rectangle $EFGH$ with base $EF$ and altitude $h$.

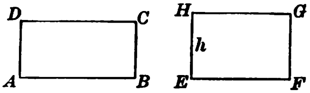


Then, $EFGH$ is the required rectangle.

**Proof.** Const., $EF : AB = AB : h$; $\therefore$ $EF \times h = \overline{AB}^2$;
that is, $EFGH \Leftrightarrow ABCD$.

**Ex. 599.** Construct a rectangle equivalent to a given rectangle and having a given side.

**Solution.** § 317, find $h$ the fourth proportional to the given side $EF$ and the base and altitude, $AB$ and $AD$ respectively, of the given rectangle $ABCD$.



With $EF$ as base and $h$ as altitude construct rectangle $EFGH$.

Then, $EFGH$ is the required rectangle.

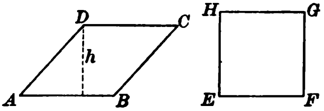
**Proof.** Const., $EF : AB = AD : h$; $\therefore$ $EF \times h = AB \times AD$;
that is, $EFGH \Leftrightarrow ABCD$.

**Ex. 600.** Construct a square equivalent to a given rhombus.

**Solution.** Draw the altitude $h$ of the given rhombus $ABCD$.

Find $EF$ the mean proportional between $AB$ and $h$.

On $EF$ as a side construct the square $EFGH$.



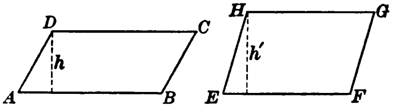
Then, $EFGH$ is the required square.

**Proof.** Proceed as in Ex. 596.

**Ex. 601.** Construct a parallelogram having a given altitude, and equivalent to a given parallelogram.

**Solution.** Draw the altitude $h$ of the given ▱ $ABCD$.

Find $EF$, the fourth proportional to the given altitude $h'$, $h$, and $AB$.



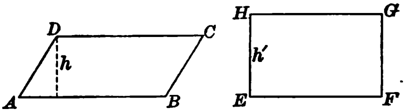
On $EF$ as base construct ▱ $EFGH$ having the altitude $h'$.

Then, $EFGH$ is the required ▱.

**Proof.** Const., $h' : h = AB : EF$; $\therefore$ $EF \times h' = AB \times h$;
that is, $EFGH \Leftrightarrow ABCD$.

**Sch.** Since the size of $\angle E$ may vary, there may be an indefinite number of parallelograms fulfilling the conditions.

Ex. **602**. Construct a rectangle having a given altitude, and equivalent to a given parallelogram.



**Solution.** Draw the altitude $h$ of the given ▱ $ABCD$.

Find $EF$, the fourth proportional to $h'$, the given altitude, $AB$, and $h$.

With $EF$ as base and altitude $h'$, construct the rectangle $EFGH$.

Then, $EFGH$ is the rectangle required.

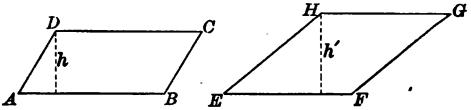
**Proof.** Proceed as in Ex. 601.

Ex. **603**. Construct a rhombus having a given side, and equivalent to a given parallelogram.

**Solution.** Draw the altitude $h$ of the given ▱ $ABCD$.

Find $h'$, the fourth proportional to the given side $EF$, $AB$, and $h$.

Erect a perpendicular to $EF$ equal to $h'$, and through its extremity draw a line parallel to $EF$; with $E$ as center and radius $EF$ describe an arc cutting this parallel in $H$; on the parallel take $HG = EF$, and draw $FG$.



Then, $EFGH$ is the required rhombus.

**Proof.** Since $HG$ is equal and parallel to $EF$,

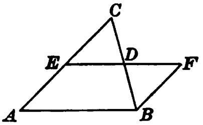
§ 150, $EFGH$ is a ▱, and, § 151, const., $EH = FG = EF = HG$;
hence, $EFGH$ is a rhombus, and, as in Ex. 601, $EFGH \approx ABCD$.

**Sch.** As in Ex. 601, there may be an indefinite number of solutions.

Ex. **604**. Construct a rhombus having a given altitude and equivalent to a given parallelogram.

**Solution.** From data find the side of the required rhombus as the altitude was found in Ex. 603, and construct the rhombus as in that exercise.

Ex. **605**. Transform a triangle into an equivalent parallelogram, whose base shall be the base of the triangle and one of whose base angles shall be equal to a base angle of the triangle.



**Solution.** Through $E$, the middle point of side $AC$ of the given $\triangle ABC$, draw $ED \parallel AB$ meeting $BC$ in $D$; produce $ED$ to $F$ making $DF = ED$; and draw $BF$.

Then, $ABFE$ is the ▱ required.

**Proof.** $EF \parallel AB$, and since, § 158, $ED = \frac{1}{2} AB$, $EF = AB$, and, § 150, $ABCD$ is a ▱.

Const., $DF = ED$, § 158, $DB = CD$, and, § 59, $\angle BDF = \angle EDC$;

hence, $\triangle DFB = \triangle EDC$;

consequently, $ABDE + \triangle DFB \approx ABDE + \triangle EDC$;

that is, $ABFE \approx \triangle ABC$.

**Ex. 606.** Construct a triangle having a given angle, and equivalent to a given parallelogram.

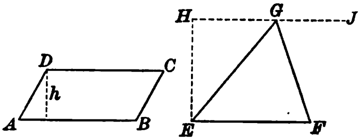
**Solution.** Draw the altitude $h$ of the given ▱ $ABCD$.

Draw $EF = AB$; at $E$ erect the $\perp$ $EH = 2h$; and draw $HJ \parallel EF$.

Construct $\angle FEG =$ the given $\angle$, the side $EG$ meeting $HJ$, as at $G$.

Draw $FG$.

Then, $\triangle EFG$ is the $\triangle$ required.



**Proof.** § 332, $\quad$ area $ABCD = AB \times h$,

and, § 335, area $\triangle EFG = \frac{1}{2} EF \times EH = \frac{1}{2} AB \times 2h = AB \times h$;

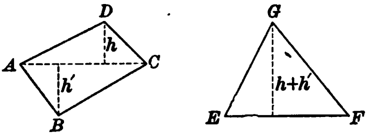
hence, $\quad \triangle EFG \Leftrightarrow ABCD$. See Sch., Ex. 518.

**Ex. 607.** Construct a triangle equivalent to a given trapezium.

**Solution.** Draw the diagonal $AC$ of the given trapezium $ABCD$ and the perpendiculars $h$ and $h$ from $D$ and $B$ respectively upon $AC$.

Draw $EF = AC$, and on $EF$ as base construct $\triangle EFG$, having its altitude $= h + h'$.

Then, $\triangle EFG$ is the $\triangle$ required.



**Proof.** Area $ABCD =$ area $\triangle ACD +$ area $\triangle ACB$

$$= \tfrac{1}{2} AC \times h + \tfrac{1}{2} AC \times h' = \tfrac{1}{2} AC(h + h');$$

but $\quad$ area $\triangle EFG = \frac{1}{2} EF(h + h') = \frac{1}{2} AC(h + h')$;

hence, $\quad \triangle EFG \Leftrightarrow ABCD$.

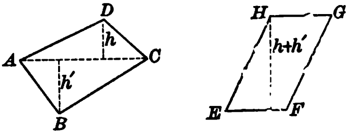
**Sch.** Since there may be an indefinite number of triangles equivalent to $EFG$, there may be an indefinite number of triangles equivalent to $ABCD$.

**Ex. 608.** Construct a parallelogram equivalent to a given trapezium.

**Solution.** Draw the diagonal $AC$ of the given trapezium $ABCD$, and the perpendiculars $h$ and $h'$, from $D$ and $B$ respectively, upon $AC$.

On $EF = \frac{1}{2} AC$ as base and with altitude $= h + h'$ construct the ▱ $EFGH$.

Then, $EFGH$ is the ▱ required.



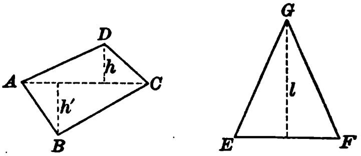
**Proof.** As in Ex. 607, $\quad$ area $ABCD = \frac{1}{2} AC(h + h')$,

and, § 332, $\quad$ area $EFGH = EF(h + h') = \frac{1}{2} AC(h + h')$,

hence, $\quad EFGH \Leftrightarrow ABCD$.

**Sch.** As in Ex. 601, there may be an indefinite number of solutions

**Ex. 609.** Construct an isosceles triangle on a given base and equivalent to a given trapezium.



**Solution.** Draw the diagonal $AC$ of the given trapezium $ABCD$, and the perpendiculars $h$ and $h'$, from $D$ and $B$ respectively, upon $AC$.

On the given base $EF$ construct an isosceles $\triangle EFG$ with altitude $l$ equal to the fourth proportional to $EF$, $AC$, and $(h + h')$.

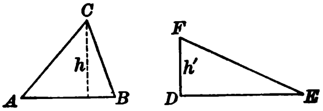
Then, $\triangle EFG$ is the $\triangle$ required.

**Proof.** Const., $EF : AC = h + h' : l$;

$\therefore$ $EF \times l = AC(h + h')$; and, Ax. 7, $\frac{1}{2} EF \times l = \frac{1}{2} AC(h + h')$;

that is, $\triangle EFG \Leftrightarrow ABCD$.

**Ex. 610.** Construct a right triangle equivalent to a given triangle, having given one of the sides about the right angle.



**Solution.** Draw the altitude $h$ of the given $\triangle ABC$ and find $h'$ the fourth proportional to the given side $DE$, $AB$, and $h$.

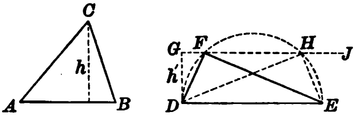
Draw $DF \perp DE$, the given side, and equal to $h'$. Draw $EF$.

Then, $\triangle DEF$ is the $\triangle$ required.

**Proof.** Const., $DE : AB = h : h'$; $\therefore$ $DE \times h' = AB \times h$,

and $\frac{1}{2} DE \times h' = \frac{1}{2} AB \times h$; that is, $\triangle DEF \Leftrightarrow \triangle ABC$.

**Ex. 611.** Construct a right triangle equivalent to a given triangle, having given the hypotenuse.



**Solution.** Draw the altitude $h$ of the given $\triangle ABC$ and find $h'$ the fourth proportional to the given hypotenuse $DE$, $AB$, and $h$.

Draw $DG \perp DE$ and equal to $h'$; draw $GJ \parallel DE$; on $DE$ as a diameter describe a semicircumference intersecting $GJ$ as at $F$ and $H$. Draw $DF$ and $EF$.

Then, $\triangle DEF$ is the $\triangle$ required.

**Proof.** § 227, $\angle DFE$ is a rt. $\angle$. Proceed as in Ex. 610.

**Sch.** $\triangle DEH$ also fulfills the conditions required.

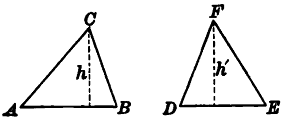
**Ex. 612.** Construct a triangle equivalent to a given triangle and having its base and altitude equal.

**Solution.** Draw the altitude $h$ of the given $\triangle ABC$, and find $h'$ the mean proportional to $AB$ and $h$.

Construct $\triangle DEF$ having its base and altitude each equal to $h$.

Then, $\triangle DEF$ is the $\triangle$ required.

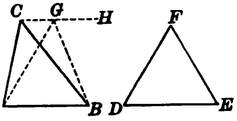
**Proof.** Proceed as in Ex. 610.



**Sch.** Any triangle on base $DE$ whose vertex lies on a line parallel to $DE$ through $F$ will fulfill the conditions, consequently, there may be an indefinite number of solutions.

**Ex. 613.** Construct an equilateral triangle equivalent to a given triangle.

**Solution.** Through $C$, the vertex of the given $\triangle ABC$, draw $CH \parallel AB$; draw $AG$ making with $AB$ an $\angle$ of 60° and meeting $CH$ as in $G$. Draw $BG$.



Find $DE$ a mean proportional between $AG$ and $AB$.

On $DE$ construct an equilateral $\triangle DEF$.

Then, $\triangle DEF$ is the $\triangle$ required.

**Proof.** In $\triangle ABG$ and $DEF$, $\angle GAB = \angle D$;

$\therefore$ § 340, $\triangle ABG : \triangle DEF = AG \times AB : DF \times DE$;

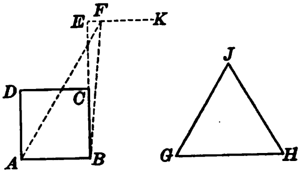
but, const., § 269, $AG \times AB = \overline{DE}^2$, and $DF \times DE = \overline{DE}^2$;

$\therefore$ $AG \times AB = DF \times DE$, and, § 341, $\triangle ABG \Leftrightarrow \triangle DEF$;

but, Ex. 481, $\triangle ABG \Leftrightarrow \triangle ABC$; hence, $\triangle DEF \Leftrightarrow \triangle ABC$.

**Ex. 614.** Construct an equilateral triangle equivalent to a given square.

**Solution.** Produce $BC$, a side of the given square $ABCD$, to $E$, making $CE = BC$; draw $EK \parallel AB$; construct $\angle BAF = 60°$, the side $AF$ meeting $EK$ in $F$; and draw $BF$.



By Ex. 613, construct equilateral

$$\triangle GHJ \Leftrightarrow \triangle ABF.$$

Then, $\triangle GHJ$ is the $\triangle$ required.

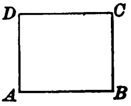
**Proof.** § 335, area $\triangle ABF = \frac{1}{2} AB \times BE = \frac{1}{2} AB \times 2\,BC = \overline{AB}^2$;

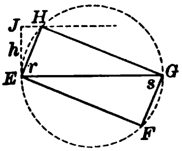
$\therefore$ $\triangle ABF \Leftrightarrow ABCD$;

but, const., $\triangle ABF \Leftrightarrow \triangle GHJ$;

hence, $\triangle GHJ \Leftrightarrow ABCD$.

**Ex. 615.** Construct a rectangle having a given diagonal, and equivalent to a given rectangle.





**Solution.** Find $h$ a fourth proportional to the given diagonal $EG$, and the base $AB$ and altitude $AD$ of the given rectangle $ABCD$.

Draw $EJ \perp EG$, and equal to $h$, and through $J$ draw a line parallel to $EG$, cutting the circumference described on $EG$ as a diameter as at $H$. Draw $EH$ and $HG$.

With $G$ as center and radius equal to $EH$ describe an arc cutting the circumference, as at $F$. Draw $EF$ and $GF$.

Then, $EFGH$ is the required rectangle.

**Proof.** § 227, $\triangle$s $EGH$ and $EFG$ are rt. $\triangle$s,
const., $EH = FG$, and $EG$ is common;
$\therefore$ § 123, $\triangle EGH = \triangle EFG$, and $\angle r = \angle s$;
hence, § 75, $EH \parallel FG$,
and, § 150, $EFGH$ is a ▱;
consequently, Ex. 104, $EFGH$ is a rectangle.
Area $\triangle EGH = \frac{1}{2} EG \times h$;
$\therefore$ area $EFGH = EG \times h$;
area $ABCD = AB \times AD$;
but, const., $EG : AB = AD : h$;
that is, $EG \times h = AB \times AD$;
hence, $EFGH \Bumpeq ABCD$.

**Sch.** The problem is impossible if the given diagonal is less than the diagonal of the given rectangle.

**Ex. 616.** Construct a rectangle having a given diagonal, and equivalent to a given square.

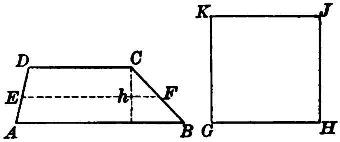
**Solution.** Find $h$ a third proportional to the given diagonal $EG$, and the side $AB$ of the given square $ABCD$.

Proceed as in Ex. 615.

**Ex. 617.** Construct a square equivalent to a given trapezoid.

**Solution.** Draw the altitude $h$, of the given trapezoid $ABCD$, and the line $EF$ joining the middle points of the non-parallel sides $AD$ and $BC$.

Find $GH$ the mean proportional between $EF$ and $h$.



On $GH$ as a side construct the square $GHJK$.

Then, $GHJK$ is the square required.

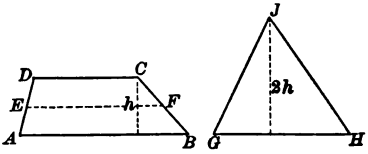
**Proof.** Const., $EF : GH = GH : h$;
$\therefore$ $\overline{GH}^2 = EF \times h$;
§§ 338, 160, area $ABCD = \frac{1}{2}(AB + DC)h = EF \times h$;
hence, $GHJK \Bumpeq ABCD$.

**Ex. 618.** Construct a triangle equivalent to a given trapezoid.

**Solution.** Draw the altitude $h$, of the given trapezoid $ABCD$, and the line $EF$ joining the middle points of the non-parallel sides $AD$ and $BC$.

With $GH = EF$ as base, and $2h$ as altitude, construct $\triangle GHJ$.

Then, $\triangle GHJ$ is the $\triangle$ required.



**Proof.** §§ 338, 160, area $ABCD = \frac{1}{2}(AB + DC)h = EF \times h$,

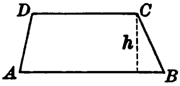
and, § 335, area $\triangle GHJ = \frac{1}{2} GH \times 2h = GH \times h = EF \times h$;

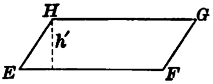
hence, $\triangle GHJ \bumpeq ABCD$.

**Sch.** As in Sch. for Ex. 607, there may be an indefinite number of $\triangle$s fulfilling the conditions.

Ex. **619.** Construct a parallelogram equivalent to a given trapezoid, and having for its base the longer base of the trapezoid.

**Solution.** Draw the altitude $h$, of the given trapezoid $ABCD$, whose parallel sides are $AB$ and $DC$.





Find $h'$ the fourth proportional to $AB$, $\frac{1}{2}(AB + DC)$, and $h$.

On $EF = AB$, as base, construct ▱ $ABCD$, having altitude $= h'$.

Then, $EFGH$ is the ▱ required.

**Proof.** Const., $AB : \frac{1}{2}(AB + DC) = h : h'$;

whence, $AB \times h' = \frac{1}{2}(AB + DC)h$;

but area $EFGH = EF \times h' = AB \times h' = \frac{1}{2}(AB + DC)h$,

and, § 338, area $ABCD = \frac{1}{2}(AB + DC)h$;

hence, $EFGH \bumpeq ABCD$.

**Sch.** As in Sch. for Ex. 601, there may be an indefinite number of solutions.

Ex. **620.** Construct a triangle equivalent to a given triangle and similar to another given triangle.

**Solution.** Proceed as in § 372.

Ex. **621.** Construct a parallelogram equivalent to the sum of two given parallelograms.

**Solution.** Denote the given ▱s by $A$ and $B$.

§ 368, construct squares equivalent to $A$ and $B$ respectively, and, § 362, construct a square $C$ equivalent to the sum of these squares.

Then, construct ▱ $EFGH \bumpeq C$ by taking some line as $EF$ for base and an altitude equal to the third proportional to $EF$ and the side of $C$.

**Proof.** $EFGH \bumpeq C$, as in Ex. 596, and, const., $A + B \bumpeq C$; hence, $EFGH \bumpeq A + B$.

**Sch.** Since $EF$ is not of definite length, there may be an indefinite number of solutions.

Ex. **622.** The area of a square is 16. Construct a square that shall be to it in the ratio of 5 to 3.

**Solution.** Proceed as in § 366, substituting ratio $5:3$ for $m:n$, and $\sqrt{16} = 4$ for the side of $A$.

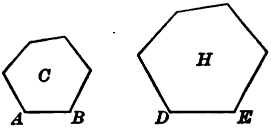
Ex. **623**. Construct a hexagon similar to a given hexagon, having its ratio to the given hexagon as 5 is to 3.

**Solution.** Denote the given hexagon by $C$ and a side of it by $AB$.

§ 366, find a line $DE$, such that the square constructed upon it shall be to the square constructed upon $AB$ as 5 is to 3.

On $DE$ as a side homologous to $AB$ construct the hexagon $H$ similar to $C$.

Then, $H$ is the required hexagon.



**Proof.** § 344, $H : C = \overline{DE}^2 : \overline{AB}^2$; but, const., $\overline{DE}^2 : \overline{AB}^2 = 5 : 3$;

hence, $H : C = 5 : 3.$

Ex. **624**. Construct a square equivalent to two thirds of a given hexagon.

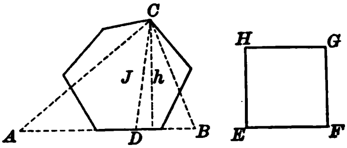
**Solution.** § 367, construct $\triangle ABC \approx J$, the given hexagon.

On $AB$ take $AD = \frac{2}{3} AB$; draw $CD$ and the altitude $h$ of $\triangle ADC$.

Find $EF$, the mean proportional between $AD$ and $\frac{1}{2} h$.

On $EF$ as a side construct square $EFGH$.

Then, $EFGH$ is the required square.



**Proof.** Const., $AD : EF = EF : \frac{1}{2} h$; $\therefore \overline{EF}^2 = \frac{1}{2} AD \times h$;

that is, $EFGH \approx \triangle ADC$; but, § 336, $\triangle ADC \approx \frac{2}{3} \triangle ABC \approx \frac{2}{3} J$;

hence, $EFGH \approx \frac{2}{3} J.$

Ex. **625**. Construct a square equivalent to the sum of a given pentagon and a given parallelogram.

**Solution.** § 369, construct the square $C \approx A$, the given pentagon, and construct the square $D \approx B$, the given parallelogram.

§ 362, construct the square $H \approx C + D$.

Then, $H$ is the required square.

**Proof.** Const., $H \approx C + D \approx A + B.$

Ex. **626**. Divide a given triangle into two equivalent parts by a line perpendicular to one side.

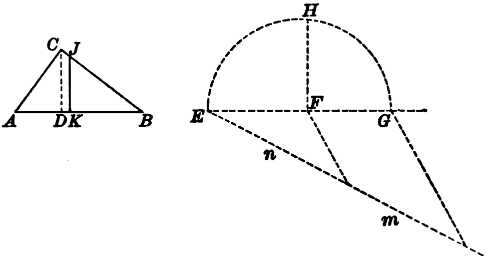
**Solution.** Draw $CD \perp AB$, the base of the given $\triangle ABC$.

Take $EF = BC$, $n = 2\,DB$, and $m = AB$.

As in § 366, find $FH$ such that $\overline{FH}^2 : \overline{EF}^2 = m : n$.

On $BC$ take $BJ = FH$, and draw $JK \perp AB$.

Then, $AKJC$ and $\triangle KBJ$ are the required parts.



**Proof.** Const., $\overline{FH}^2 : \overline{EF}^2 = m : n$; that is, § 275, $\overline{EF}^2 : \overline{FH}^2 = n : m$.

Then, since, § 302, $\triangle$s $DBC$ and $KBJ$ are similar,

§ 342, $\triangle DBC : \triangle KBJ = \overline{BC}^2 : \overline{BJ}^2 = \overline{EF}^2 : \overline{FH}^2 = n : m$;

that is, $\triangle DBC : \triangle KBJ = 2\,DB : AB$;

but, § 336, $\triangle ABC : \triangle DBC = AB : DB$;

hence, § 287, $\triangle ABC : \triangle KBJ = 2 : 1$;

that is, $AKJC \Leftrightarrow \triangle KBJ$.

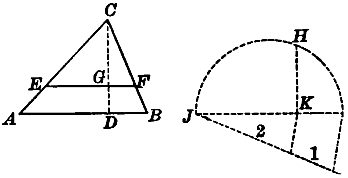
**Ex. 627.** Divide a given triangle into two equivalent parts by a line parallel to one side.

**Solution.** Draw $CD \perp AB$, the base of the given $\triangle ABC$, and take $JK = CD$.

As in § 366, find $HK$, such that

$$\overline{HK}^2 : \overline{JK}^2 = 1 : 2.$$

On $CD$ lay off $CG = HK$, and through $G$ draw $EF \parallel AB$ terminating in $AC$ and $BC$.



Then, $ABFE$ and $\triangle EFC$ are the parts required.

**Proof.** Const., $\overline{HK}^2 : \overline{JK}^2 = 1 : 2$; that is, § 275, $\overline{JK}^2 : \overline{HK}^2 = 2 : 1$.

Then, since, § 301, $\triangle$s $ABC$ and $EFC$ are similar,

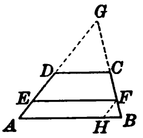
§ 343, $\triangle ABC : \triangle EFC = \overline{CD}^2 : \overline{CG}^2$;

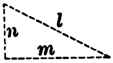
but $\overline{CD}^2 : \overline{CG}^2 = \overline{JK}^2 : \overline{HK}^2 = 2 : 1$;

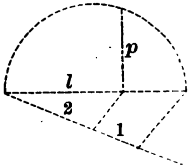
hence, $\triangle ABC : \triangle EFC = 2 : 1$;

that is, $ABFE \Leftrightarrow \triangle EFC$.

**Ex. 628.** Bisect a given trapezoid by a line parallel to the bases.







**Solution.** Draw a rt. $\triangle$ with sides $m$ and $n$ about the rt. $\angle$ equal respectively to the bases $AB$ and $DC$ of the given trapezoid $ABCD$. Denote the hypotenuse by $l$.

As in § 366, find $p$ such that $p^2 : l^2 = 1 : 2$.

On $AB$ take $AH = p$; draw $HF \parallel AD$ and $EF \parallel AB$ terminating in the sides $AD$ and $BC$.

Then, $EF$ bisects $ABCD$.

**Proof.** Produce $AD$ and $BC$ to meet, as at $G$.

§ 301, $\triangle$s $ABG$, $EFG$, and $DCG$ are similar;

hence, $$\triangle EFG : \triangle DCG = \overline{EF}^2 : \overline{DC}^2; \quad (1)$$

and $$\triangle ABG : \triangle DCG = \overline{AB}^2 : \overline{DC}^2. \quad (2)$$

(1) by § 277, $$\triangle EFG - \triangle DCG : \triangle DCG = \overline{EF}^2 - \overline{DC}^2 : \overline{DC}^2; \quad (3)$$

but $EF = AH = p$; $\therefore \overline{EF}^2 = p^2 = \frac{1}{2} l^2 = \frac{1}{2}(m^2 + n^2) = \frac{1}{2}(\overline{AB}^2 + \overline{DC}^2)$;

$\therefore$ substituting in (3),

$$\triangle EFG - \triangle DCG : \triangle DCG = \tfrac{1}{2}(\overline{AB}^2 - \overline{DC}^2) : \overline{DC}^2; \quad (4)$$

(2) by § 277, $$\triangle ABG - \triangle DCG : \triangle DCG = \overline{AB}^2 - \overline{DC}^2 : \overline{DC}^2; \quad (5)$$

from (4) and (5),

$$\triangle EFG - \triangle DCG : \triangle ABG - \triangle DCG = \tfrac{1}{2}(\overline{AB}^2 - \overline{DC}^2) : (\overline{AB}^2 - \overline{DC}^2). \quad (6)$$

But $\triangle EFG - \triangle DCG$ is trapezoid $EFCD$,

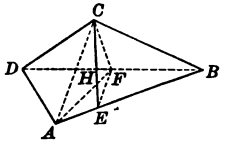
and $\triangle ABG - \triangle DCG$ is trapezoid $ABCD$;

$\therefore$ substituting in (6) and dividing the second couplet by $(\overline{AB}^2 - \overline{DC}^2)$,

$$EFCD : ABCD = \tfrac{1}{2} : 1;$$

that is, $EFCD \approx \frac{1}{2} ABCD$, or $EF$ bisects $ABCD$.

**Ex. 629.** Bisect a given quadrilateral by a line drawn from one of the vertices.



**Solution.** Draw the diagonals $AC$ and $BD$ of the given quadrilateral $ABCD$.

Find $F$ the middle point of $BD$ and draw $AF$ and $CF$; draw $FE \parallel CA$ meeting the perimeter as at $E$; and draw $CE$ from the given vertex $C$.

Then, $CE$ bisects $ABCD$.

**Proof.** § 336, $\triangle DFC \approx \triangle FBC$, and $\triangle AFD \approx \triangle ABF$;

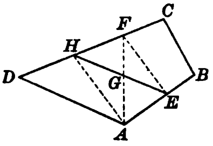
hence, $$AFCD \approx ABCF.$$

Ex. 481. $\triangle AFC \Leftrightarrow \triangle AEC$, and taking from each member of this equivalence the common part $\triangle AHC$, $\triangle CHF \Leftrightarrow \triangle AEH$;

consequently, $AECD \Leftrightarrow AFCD \Leftrightarrow \frac{1}{2} ABCD$;

that is, $CE$ bisects $ABCD$.

Ex. **630**. Bisect a given quadrilateral by a line drawn from any point in its perimeter.



**Solution.** By Ex. 629, bisect the given quadrilateral $ABCD$ by the line $AF$ from the vertex $A$.

Draw $EF$ from $E$ the given point in the perimeter; draw $AH \parallel EF$ meeting $DC$ as in $H$; and draw $EH$.

Then, $EH$ bisects $ABCD$.

**Proof.** Ex. 559, $\triangle AGE \Leftrightarrow \triangle HGF$;

hence, $AGHD + \triangle AGE \Leftrightarrow AGHD + \triangle HGF$;

consequently, $AEHD \Leftrightarrow AFD \Leftrightarrow \frac{1}{2} ABCD$;

that is, $EH$ bisects $ABCD$.

Ex. **637**. The three sides of a triangle are 58 ft., 51 ft., and 41 ft. in length. What is the area of the triangle?

**Solution.** Ex. 632, $A = \sqrt{75(75-58)(75-51)(75-41)}$

$$= \sqrt{75 \times 17 \times 24 \times 34} = \sqrt{1040400} = 1020 \text{ sq. ft.}$$

Ex. **638**. Find the altitude on each of the sides of a triangle whose sides are respectively 7 in., 9 in., and 11 in.

**Solution.** Ex. 631, $h$ (on side 7 in.) $= \frac{2}{7}\sqrt{\frac{27}{2} \times \frac{13}{2} \times \frac{9}{2} \times \frac{5}{2}}$

$$= \frac{2}{7} \times \frac{1}{4}\sqrt{27 \times 13 \times 9 \times 5} = \frac{1}{14}\sqrt{15795} = \frac{1}{14} \times 125.67+ = 8.97+ \text{ in.}$$

$$h \text{ (on side 9 in.)} = \frac{2}{9} \times \frac{1}{4}\sqrt{15795} = \frac{1}{18} \times 125.67+ = 6.98+ \text{ in.}$$

$$h \text{ (on side 11 in.)} = \frac{2}{11} \times \frac{1}{4}\sqrt{15795} = \frac{1}{22} \times 125.67+ = 5.71+ \text{ in.}$$

Ex. **639**. If the sides of a triangle are respectively $4^m$, $6^m$, and $8^m$ long, what are its three medians?

**Solution.** Ex. 633, $m$ (to side $4^m$) $= \frac{1}{2}\sqrt{2(6^2 + 8^2) - 4^2}$

$$= \frac{1}{2}\sqrt{184} = \frac{1}{2} \times 13.56+ = 6.78+^m$$

$$m \text{ (to side } 6^m) = \frac{1}{2}\sqrt{2(4^2 + 8^2) - 6^2} = \frac{1}{2}\sqrt{124} = \frac{1}{2} \times 11.13+ = 5.56+^m$$

$$m \text{ (to side } 8^m) = \frac{1}{2}\sqrt{2(4^2 + 6^2) - 8^2} = \frac{1}{2}\sqrt{40} = \frac{1}{2} \times 6.32+ = 3.16+^m.$$

Ex. **640**. What is the area of a triangle, if the radius of the circumscribing circle is $6.196^m$ and the sides of the triangle are respectively $9^m$, $6^m$, and $12^m$ in length?

**Solution.** Ex. 635, $A = \frac{9 \times 6 \times 12}{4 \times 6.196} = 26.14+^{\text{sq m}}$.

**Ex. 641.** The sides of a triangle are respectively 12 in., 11 in., and 9 in. in length. Find the radius of the circumscribing circle.

**Solution.** Ex. 636, $r = \dfrac{12 \times 11 \times 9}{4\sqrt{16 \times 4 \times 5 \times 7}} = 6.275+$ in.

**Ex. 642.** The sides of a triangle are respectively $30^{\text{dm}}$, $50^{\text{dm}}$, and $70^{\text{dm}}$. Find the length of the three angle bisectors.

**Solution.** Ex. 634,

$$k \text{ (of } \angle \text{ opp. side } 30^{\text{dm}}) = \frac{2}{50 + 70}\sqrt{50 \times 70 \times 75 \times 45}$$
$$= \tfrac{1}{60} \times 3436.93 = 57.28+^{\text{dm}}.$$

$$k \text{ (of } \angle \text{ opp. side } 50^{\text{dm}}) = \frac{2}{30 + 70}\sqrt{30 \times 70 \times 75 \times 25}$$
$$= \tfrac{1}{50} \times 1984.31 = 39.68+^{\text{dm}}.$$

$$k \text{ (of } \angle \text{ opp. side } 70^{\text{dm}}) = \frac{2}{30 + 50}\sqrt{30 \times 50 \times 75 \times 5}$$
$$= \tfrac{1}{40} \times 750 = 18.75^{\text{dm}}.$$

**Ex. 643.** If two sides and one of the diagonals of a parallelogram are respectively 12 in., 15 in., and 18 in., what is the length of the other diagonal? What is the area of the parallelogram?

**Solution.** Let $ABCD$ be the ▱ whose diagonal $AC = 18$ in., side $AB = 15$ in., and side $BC = 12$ in.

Since, § 154, $AC$ and $BD$ bisect each other,

Ex. 633, $BD = 2m = \sqrt{2(12^2 + 15^2) - 18^2} = \sqrt{414} = 20.34+$ in.

Ex. 632, area $ABCD = 2\triangle ABC = 2\sqrt{\frac{45}{2} \times \frac{21}{2} \times \frac{15}{2} \times \frac{9}{2}}$

$= \frac{1}{2}\sqrt{45 \times 21 \times 15 \times 9} = 178.58+$ sq. in.

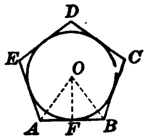
## BOOK VI

**Ex. 644.** The apothem of a regular pentagon is $41^{\text{cm}}$ and a side is $6^{\text{dm}}$. Find the perimeter of a regular pentagon whose apothem is $82^{\text{cm}}$.

**Solution.** The perimeter of the polygon whose side is given $= 6^{\text{dm}} \times 5 = 30^{\text{dm}}$.

Then, § 387, $Q : 30 = 82 : 41$; whence, $Q = \dfrac{30 \times 82}{41} = 60^{\text{dm}}$.

**Ex. 645.** The sides of a regular circumscribed polygon are bisected at the points of tangency.



**Data:** Regular polygon $ABCDE$ circumscribed about the circle whose center is $O$.

**To prove** sides $AB$, $BC$, $CD$, etc., bisected at the points of tangency.

**Proof.** Draw $OA$, $OB$, and draw $OF$ to the point of tangency of $AB$.

Then, § 205, $OF \perp AB$.

In rt. $\triangle$ $AFO$ and $BFO$, $OA = OB$, and $OF$ is common;

∴ § 123, $\triangle AFO = \triangle BFO$, and $AF = BF$.

Similarly, $BC$, $CD$, etc., are bisected at the points of tangency.

**Ex. 646.** The angle at the center of a regular polygon is the supplement of the angle of the polygon.

**To prove** (See Fig. for Ex. 645) $\angle AOB$ the sup. of $\angle BAE$.

**Proof.** § 110, $\angle AOB + \angle OAB + \angle OBA = 2$ rt. $\angle$s. (1)

Now, § 116, $\angle OBA = \angle OAB$, and, § 382, $\angle OAE = \angle OAB$;

∴ $\angle OBA = \angle OAE$;

substituting in (1), $\angle AOB + (\angle OAB + \angle OAE) = 2$ rt. $\angle$s;

that is, $\angle AOB + \angle BAE = 2$ rt. $\angle$s;

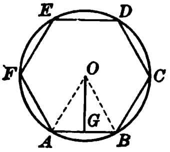
consequently, § 32, $\angle AOB$ is the sup. of $\angle BAE$.

**Ex. 647.** Find the angle and the angle at the center of a regular dodecagon.

**Solution.** § 383, the $\angle$ at the center $= \frac{1}{12}$ of $360° = 30°$.

Ex. 646, the $\angle$ of the polygon is the sup. of $30° = 150°$.

**Ex. 648.** If the radius of a regular inscribed hexagon is $r$, prove that its side $= r$, and its apothem $= \frac{1}{2} r\sqrt{3}$.



**Data:** Regular hexagon $ABCDEF$ whose radius is $r$ and apothem $OG$, inscribed in the ⊙ whose center is $O$.

**To prove** any side, as $AB = r$, and $OG = \frac{1}{2} r\sqrt{3}$.

**Proof.** Draw the radii $OA$ and $OB$.

Since arc $AB = \frac{1}{6}$ of the circumference,

$\angle O = \frac{1}{6}$ of $360° = 60°$;

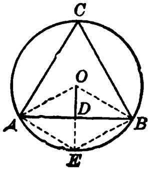
hence, $\angle OAB + \angle OBA = 180° - 60° = 120°$; but, since $OA = OB$,

§ 116, $\angle OAB = \angle OBA = \frac{1}{2}$ of $120° = 60°$;

hence, § 119, $\triangle ABO$ is equilateral, and $AB = AO = r$.

Also, $OG = \sqrt{\overline{OA}^2 - \overline{AG}^2} = \sqrt{r^2 - (\frac{1}{2} r)^2} = \frac{1}{2} r\sqrt{3}$.

**Ex. 649.** If the radius of an inscribed equilateral triangle is $r$, prove that its side $= r\sqrt{3}$, and its apothem $= \frac{1}{2} r$.



**Data:** Equilateral $\triangle ABC$ whose radius is $r$ and apothem $OD$, inscribed in the ⊙ whose center is $O$.

**To prove** any side, as $AB = r\sqrt{3}$, and $OD = \frac{1}{2} r$.

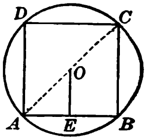
**Proof.** Produce $OD$ to meet the arc, as at $E$, and draw $AO$, $AE$, $BO$, and $BE$.

Since, § 200, $E$ is the middle point of arc $AB$,
arc $AE = \frac{1}{2}$ arc $AB = \frac{1}{2}$ of $\frac{1}{3}$ of the circumference, or $\frac{1}{6}$ of the circumference;
consequently, $AE$ is the side of a regular inscribed hexagon;
similarly, $BE$ is the side of a regular inscribed hexagon;
hence, Ex. 648, $AO = AE = BO = BE$;
∴ § 106, $AB$ and $OE$ bisect each other at $D$.

Then, $\overline{AD}^2 = \overline{AO}^2 - \overline{OD}^2$, or $(\frac{1}{2} AB)^2 = r^2 - (\frac{1}{2} r)^2$;
that is, $\frac{\overline{AB}^2}{4} = r^2 - \frac{r^2}{4}$; whence, $\overline{AB}^2 = 4r^2 - r^2 = 3r^2$, and $AB = r\sqrt{3}$.

Also, $OD = \frac{1}{2} OE = \frac{1}{2} r$.

Ex. **650**. If the radius of an inscribed square is $r$, prove that its side $= r\sqrt{2}$, and its apothem $= \frac{1}{2} r\sqrt{2}$.



**Data:** Square $ABCD$ whose radius is $r$ and apothem $OE$, inscribed in the ⊙ whose center is $O$.

**To prove** any side, as $AB = r\sqrt{2}$, and $OE = \frac{1}{2} r\sqrt{2}$.

**Proof.** Draw the diagonal $AC$. § 351, $\frac{AC}{AB} = \sqrt{2}$;
that is, $\frac{2\,AO}{AB}$, or $\frac{2\,r}{AB} = \sqrt{2}$; whence, $AB = r\sqrt{2}$.
§ 71, $EO \parallel BC$ and $O$ is the middle point of $AC$;
hence, § 158, $OE = \frac{1}{2} BC = \frac{1}{2} AB = \frac{1}{2} r\sqrt{2}$.

Ex. **651**. The radius of a circle being $r$, find the area of an inscribed equilateral triangle.

**Solution.** Ex. 649, a side $= r\sqrt{3}$, the perimeter $= 3\,r\sqrt{3}$, and the apothem $= \frac{1}{2} r$;
hence, § 389, area $= \frac{1}{2} \times \frac{1}{2} r \times 3\,r\sqrt{3} = \frac{3}{4} r^2\sqrt{3}$.

Ex. **652**. The radius of a circle being $r$, find the area of an inscribed square.

**Solution.** Ex. 650, a side $= r\sqrt{2}$; hence, area $= (r\sqrt{2})^2 = 2\,r^2$.

Ex. **653**. Find the area of a regular hexagon whose side is 10 ft.

**Solution.** Ex. 648, the apothem $= \frac{1}{2} r\sqrt{3} = 5\sqrt{3}$ ft., and the perimeter $= 10 \times 6 = 60$ ft.;
hence, § 389, area $= \frac{1}{2} \times 60 \times 5\sqrt{3} = 150\sqrt{3} = 259.8+$ sq. ft.

Ex. **654**. If the circumferences of two circles are $314.16^{\text{cm}}$ and $157.08^{\text{cm}}$ respectively, and the radius of the first is $50^{\text{cm}}$, what is the radius of the second? What is the area of the first?

**Solution.** (1) Denote the required radius by $r$.

Then, § 394, $$314.16 : 157.08 = 50 : r;$$

whence, $$r = \frac{157.08 \times 50}{314.16} = 25^{\text{cm}}.$$

(2) § 397, area of first $\odot = \frac{1}{2} \times 314.16 \times 50 = 7854^{\text{sq cm}}$.

Ex. **655**. Find the circumference and area of a circle whose radius is $2.5^{\text{dm}}$.

**Solution.** (1) § 395, $C = 2\pi R = 2 \times 3.1416 \times 2.5 = 15.708^{\text{dm}}$.

(2) § 398, area $= \pi R^2 = 3.1416 \times 2.5^2 = 19.635^{\text{sq dm}}$.

Ex. **656**. What is the ratio of the radii of two circles, if the area of one circle is twice that of the other?

**Solution.** Denote the radii by $R$ and $R'$.

Then, § 399, $$2 : 1 = R^2 : R'^2;$$

hence, § 288, $$\sqrt{2} : \sqrt{1} = R : R', \text{ or } R : R' = \sqrt{2} : 1.$$

Ex. **657**. What is the area of a sector whose arc is $\frac{1}{6}$ of the circumference, if the radius of the circle is $18^{\text{dm}}$?

**Solution.** § 395, $C = 2\pi R$; $\therefore$ the arc of the sector $= \frac{1}{6}$ of $2\pi R = \frac{1}{3}\pi R$, and, § 400, the area of the sector $= \frac{1}{2} \times \frac{1}{3}\pi R \times R = \frac{1}{6}\pi R^2 = \frac{1}{6} \times 3.1416 \times 18^2 = 169.6464^{\text{sq dm}}$.

Ex. **658**. If the radius of a circle is 63 in., how long is the arc of a sector whose angle is 45°?

**Solution.** The arc is such a part of the circumference as 45° is of 360°, or $\frac{45}{360} = \frac{1}{8}$ of the circumference.

Now, § 395, $C = 2\pi R = 2 \times 3.1416 \times 63 = 395.8416$ in.;

hence, the arc of the sector $= \frac{1}{8}$ of $395.8416 = 49.4802$ in.

Ex. **659**. Calling the equatorial radius of the earth 3962.8 miles, what is the length of a degree on the equator?

**Solution.** § 395, $C = 2\pi R = 2 \times 3.1416 \times 3962.8 = 24899.06496$ mi.

The length of $1° = \frac{1}{360}$ of 24899.06496 mi. $= 69.16+$ mi.

Ex. **660**. Find the radius of a circle whose area is $6^{\text{sq m}}$.

**Solution.** § 398, $$\text{area} = \pi R^2 = 6;$$

whence, $$R^2 = \frac{6}{3.1416} = 1.9098+, \text{ and } R = \sqrt{1.9098+} = 1.38+^{\text{m}}.$$

Ex. **661**. Find the circumference of a circle whose area is 100 sq. in.

**Solution.** § 398, area $= \pi R^2 = 100$; whence, $R^2 = \frac{100}{\pi}$, and $R = \frac{10}{\sqrt{\pi}}$.

Then, § 395, $C = 2\pi R = 2\pi \times \frac{10}{\sqrt{\pi}} = 20\sqrt{\pi} = 20 \times \sqrt{3.1416} = 35.45+$ in.

**Ex. 662.** What is the area of a circle whose circumference is 100 ft.?

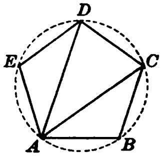
**Solution.** § 395, $C = 2\pi R = 100$; whence, $R = \frac{100}{2\pi}$.

Then, § 398, area $= \pi R^2 = \pi\left(\frac{100}{2\pi}\right)^2 = \frac{100^2}{4\pi} = \frac{2500}{3.1416} = 795.77+$ sq. ft.

**Ex. 663.** What is the area of a circle circumscribed about a square whose side is $a$?

**Solution.** Since the diagonal of the square is the diametet of the ⊙, the diameter $= \sqrt{a^2 + a^2} = a\sqrt{2}$, and the radius $= \frac{a}{2}\sqrt{2} = \frac{a}{\sqrt{2}}$; hence, § 398, area of ⊙ $= \pi R^2 = \pi\left(\frac{a}{\sqrt{2}}\right)^2 = 1.5708\,a^2$.

**Ex. 664.** The diagonals drawn from a vertex of a regular pentagon to the opposite vertices trisect the angle at that vertex.



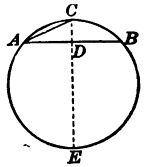
**Data:** Regular pentagon *ABCDE* and the diagonals *AC* and *AD*.

**To prove** *AC* and *AD* trisect $\angle BAE$.

**Proof.** Circumscribe a circle about *ABCDE*.

§ 374, $BC = CD = DE$; $\therefore$ § 196, arc $BC =$ arc $CD =$ arc $DE$; hence, § 225, $\angle BAC = \angle CAD = \angle DAE$; that is, *AC* and *AD* trisect $\angle BAE$.

**Ex. 665.** If the chord of an arc is $72^{\text{dm}}$, and the chord of half that arc $36.9^{\text{dm}}$, what is the diameter of the circle?



**Solution.** Let chord $AB = 72^{\text{dm}}$, and *AC*, the chord of half the arc *AB*, be $36.9^{\text{dm}}$.

Draw the diameter *CE*, cutting *AB* as at *D*.

Then, Ex. 188, $CE \perp AB$ and *D* is the middle point of *AB*.

§ 350, $CD = \sqrt{\overline{AC}^2 - \overline{AD}^2} = \sqrt{36.9^2 - 36^2} = \sqrt{65.61} = 8.1^{\text{dm}}$,

and, § 314, $DE : AD = AD : CD$; that is, $DE : 36 = 36 : 8.1$;

whence, $$DE = \frac{36 \times 36}{8.1} = 160^{\text{dm}}.$$

Then, $$CE = CD + DE = 8.1^{\text{dm}} + 160^{\text{dm}} = 168.1^{\text{dm}}.$$

**Ex. 666.** The chord of an arc is 24 in., and its altitude is 9 in. What is the diameter of the circle?

**Solution.** (See Fig. for Ex. 665.) Let chord $AB = 24$ in.

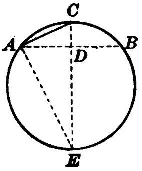
Draw the diameter $CE \perp AB$, intersecting *AB* as at *D*.

Then, *CD* is the altitude of the arc and is equal to 9 in.

§ 314, $CD : AD = AD : DE$; that is, $9 : 12 = 12 : DE$;

whence, $DE = \frac{12 \times 12}{9} = 16$ in.; hence, $CE = CD + DE = 9$ in. $+ 16$ in $= 25$ in.

**Ex. 667.** The chord of half an arc is $12^m$, and the radius of the circle is $18^m$. What is the altitude of the arc?



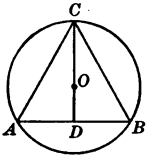
**Solution.** Let $AC = 12^m$ be the chord of half the arc $AB$. Draw the chord $AB$ and the diameter $CE \perp AB$ intersecting $AB$ as at $D$; draw $AE$.

Since, § 227, $\angle EAC$ is a rt. $\angle$, and since $AD \perp CE$,

§ 313, $CD : AC = AC : CE$; that is, $CD : 12 = 12 : 18 \times 2$;

whence, $$CD = \frac{12 \times 12}{18 \times 2} = 4^m.$$

**Ex. 668.** The altitude of an equilateral triangle is equal to one and a half times the radius of the circumscribed circle.



**Data:** Equilateral $\triangle ABC$, and its altitude $CD$; also the circumscribed circle whose center is $O$, and radius $r$.

**To prove** $CD = 1\frac{1}{2} r$.

**Proof.** § 122, $CD$ bisects $AB$; hence, § 199, $CD$ passes through $O$.

Then, $CO = r$, and, Ex. 649, $OD = \frac{1}{2} r$;

hence, $CD = CO + OD = r + \frac{1}{2} r = 1\frac{1}{2} r$.

**Ex. 669.** Find the central angle subtended by an arc whose length is equal to the radius of the circle.

**Solution.** § 395, $C = 2\pi R$; then, $R = \frac{1}{2\pi} \times C$;

hence, an arc equal to $R$ subtends a central $\angle$ which is

$$\frac{1}{2\pi} \text{ of } 360° = \frac{1}{2 \times 3.1416} \text{ of } 360° = 57.29°+.$$

Prop. XIII, page 227.

**Proof.** § 227, the $\angle$s of $ABCD$ are rt. $\angle$s.

§ 194, arc $AD$ = arc $DB$ = etc.;

∴ § 196, $AD = DB$ = etc.

Hence, § 143, $ADBC$ is a square.

Prop. XIV, page 227.

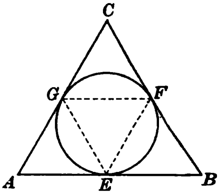
**Proof.** Draw the radii $OA$ and $OB$ to the extremities of any side as $AB$. Then, $\triangle ABO$ is equilateral, and, consequently, § 117, equiangular;

∴ $\angle O = \frac{1}{3}$ of 2 rt. $\angle$s, or $\frac{1}{6}$ of 4 rt. $\angle$s,

and the arc $AB$ is $\frac{1}{6}$ of the circumference.

Hence, the chord $AB$, which subtends the arc $AB$, is a side of a regular inscribed hexagon; that is, $ABCDEF$ is a regular inscribed hexagon.

**Ex. 670.** To circumscribe an equilateral triangle about a circle.



**Solution.** §§ 404, 407, inscribe an equilateral $\triangle EFG$.

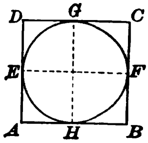
Through the vertices $E$, $F$, and $G$ draw tangents to the circle to intersect as in $A$, $B$, and $C$.

Then, $\triangle ABC$ is the $\triangle$ required.

**Proof.** § 196, arc $EF =$ arc $FG =$ arc $EG$; hence, § 376, $\triangle ABC$ is a regular circumscribed $\triangle$; that is, $\triangle ABC$ is a circumscribed equilateral $\triangle$.

**Ex. 671.** To circumscribe a square about a circle.

**Solution.** Draw any two diameters at rt. $\angle$s to each other, as $EF$ and $GH$.

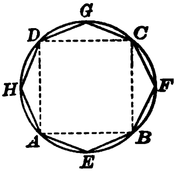


Through the points $E$, $H$, $F$, and $G$ draw tangents to the circle intersecting as in $A$, $B$, $C$, and $D$.

Then, $ABCD$ is the required square.

**Proof.** § 194, arc $EH =$ arc $HF =$ arc $FG =$ arc $EG$; hence, § 376, $ABCD$ is a regular circumscribed quadrilateral; $\therefore$ § 374, the sides of $ABCD$ are equal, and its $\angle$s are equal, and since, § 166, the sum of its $\angle$s $= 4$ rt. $\angle$s, each $\angle$ is a rt. $\angle$; hence, $ABCD$ is a circumscribed square.

**Ex. 672.** To inscribe a regular octagon in a circle.

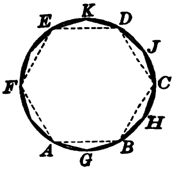


**Solution.** § 403, inscribe the square $ABCD$ in the given $\odot$.

Bisect the arcs $AB$, $BC$, $CD$, and $AD$ at $E$, $F$, $G$, and $H$, respectively; draw $AE$, $EB$, $BF$, $FC$, etc.

Then, § 408, $AEBFC$ etc., is a regular inscribed octagon.

**Ex. 673.** To inscribe a regular dodecagon in a circle.



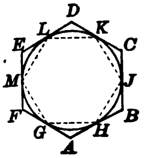
**Solution.** § 404, inscribe a regular hexagon $ABCDEF$ in the given $\odot$.

Bisect the arcs $AB$, $BC$, $CD$, etc., at $G$, $H$, $J$, etc., respectively, and draw $AG$, $GB$, $BH$, $HC$, $CJ$, etc.

Then, § 408, $AGBHCJ$ etc., is a regular inscribed dodecagon.

**Ex. 674.** To circumscribe a regular hexagon about a circle.

**Solution.** § 404, inscribe a regular hexagon $GHJKLM$ in the given $\odot$.



Through the vertices $G$, $H$, $J$, etc., draw tangents to the $\odot$ intersecting, as in $A$, $B$, $C$, etc.

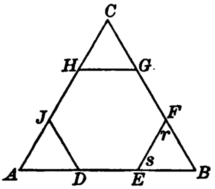
Then, $ABCDEF$ is the required hexagon.

**Proof.** § 196, arc $GH =$ arc $HJ =$ arc $JK =$ etc.; hence, § 376, $ABCDEF$ is a regular circumscribed hexagon.

**Ex. 675.** To circumscribe a regular octagon about a circle.

**Solution.** Ex. 672, inscribe a regular octagon in the given $\odot$, and proceed as in Ex. 674.

**Ex. 676.** To inscribe a regular hexagon in an equilateral triangle.



**Solution.** Trisect the sides $AB$, $BC$, and $AC$ of the given $\triangle ABC$, at $D$ and $E$, $F$ and $G$, $H$ and $J$, respectively.

Draw $EF$, $HG$, and $DJ$.

Then, $DEFGHJ$ is the required hexagon.

**Proof.** § 291, $EF \parallel AC$;

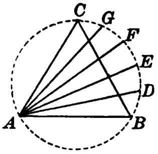
$\therefore$ § 76, $\angle s = \angle A$, and $\angle r = \angle C$,

and since, § 117, $\triangle ABC$ is equiangular, $\triangle EBF$ is equiangular; consequently, § 119, $\triangle EBF$ is equilateral, and $EF = EB = \frac{1}{3}$ of a side of $\triangle ABC$; similarly, $HG$ and $DJ$ are each equal to $\frac{1}{3}$ of a side of $\triangle ABC$; and, const., $DE$, $FG$, and $HJ$ are each equal to $\frac{1}{3}$ of a side of $\triangle ABC$; consequently, $DEFGHJ$ is equilateral.

§ 54, the $\angle$s of $DEFGHJ$ are equal.

Hence, § 374, $DEFGHJ$ is a regular hexagon.

**Ex. 677.** To divide an angle of an equilateral triangle into five equal parts.



**Solution.** Circumscribe a circle about the given $\triangle ABC$.

§ 406, find the side of a regular inscribed pentadecagon, and with it as a radius and $B$ as center describe an arc cutting the circumference of the $\odot$, as at $D$; similarly, determine the points $E$, $F$, and $G$.

From the vertex of the given $\angle A$ draw $AD$, $AE$, $AF$, and $AG$.

Then, these lines divide $\angle A$ into five equal parts.

**Proof** Arc $BC = \frac{1}{3}$ of the circumference,

and arc $BG = \frac{4}{15}$ of the circumference.;

$\therefore$ arc $GC$ = arc $BC$ − arc $BG = \frac{1}{3} - \frac{4}{15}$, or $\frac{1}{15}$ of the circumference;

hence, arc $BD$ = arc $DE$ = arc $EF$ = arc $FG$ = arc $GC$;

consequently, § 225, $\angle BAD = \angle DAE = \angle EAF = \angle FAG = \angle GAC$.

**Ex. 678.** The segment of a circle is equal to $\frac{3}{5}$ of a similar segment. What is the ratio of their radii?

**Solution.** The segments are to each other as $5:3$; hence, denoting their radii by $R$ and $R'$, respectively,

§ 402, $R^2 : R'^2 = 5 : 3$;

consequently, § 288, $R : R' = \sqrt{5} : \sqrt{3}$.

Ex. **679**. How many degrees are there in an arc 18 in. long on a circumference whose radius is 5 ft.?

**Solution.** $C = 2\pi R = 2 \times 3.1416 \times 5 = 31.416$ ft.

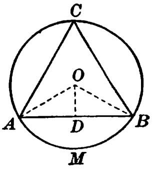
Since 18 in. $= 1\frac{1}{2}$ ft., the given arc $= \frac{1\frac{1}{2}}{31.416}$ of $360° = 17.188+°$.

Ex. **680**. The radii of two similar segments are as $3:5$. What is the ratio of their areas?

**Solution.** Denote their areas by $A$ and $B$, respectively.

Then, § 402, $A : B = 3^2 : 5^2 = 9 : 25$.

Ex. **681**. In a circle 3 ft. in diameter an equilateral triangle is inscribed. What is the area of a segment without the triangle.



**Solution.** Let $ABC$ be an equilateral $\triangle$ inscribed in a $\odot$ whose diameter is 3 ft.

Draw $OA$, $OB$, and the apothem $OD$.

§ 400, area sector $= \frac{1}{2}$ arc $AMB \times$ radius of the $\odot$; radius $= \frac{1}{2}$ of 3 ft. $= 1\frac{1}{2}$ ft., and arc $AMB = \frac{1}{3}$ circum. $= \frac{1}{3}$ of $2\pi\frac{3}{2} = 3.1416$; consequently, area of sector $= \frac{1}{2} \times 3.1416 \times \frac{3}{2} = 2.3562$ sq. ft.

Area $\triangle ABO = \frac{1}{2} AB \times OD$; but, Ex. 649, $AB = r\sqrt{3} = \frac{3}{2}\sqrt{3}$ ft., and $OD = \frac{1}{2} r = \frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$ ft.; hence, area $\triangle ABO = \frac{1}{2} \times \frac{3}{2}\sqrt{3} \times \frac{3}{4} = .97425$ sq. ft.; consequently, area seg. $AMB = 2.3562$ sq. ft. $- .97425$ sq. ft. $= 1.38195$ sq. ft.

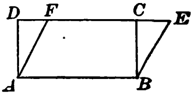
Ex. **682**. Two chords drawn from the same point in a circumference to the extremities of a diameter of a circle are 6 in. and 8 in., respectively. What is the area of the circle?

**Solution.** § 227, the $\angle$ included by these chords is a rt. $\angle$ and they form with the diameter a rt. $\triangle$.

Then, § 349, the diameter $= \sqrt{6^2 + 8^2} = 10$, and $R = 5$;

hence, § 398, area $= \pi R^2 = 3.1416 \times 5^2 = 78.54$ sq. in.

Ex. **683**. Of all equivalent parallelograms having equal bases, the rectangle has the least perimeter.



**Data**: Rectangle $ABCD$ and any equivalent $\square$, as $ABEF$, having its bases equal to those of $ABCD$.

**To prove** perimeter of $ABCD <$ perimeter of $ABEF$.

**Proof.** Since $ABCD$ and $ABEF$ are equivalent and on the same base, their altitudes are equal, and $DC$ and $FE$ are in the same straight line.

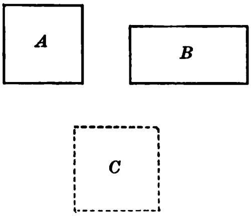
§ 61, $BC < BE$, and $AD < AF$;

$\therefore BC + AD < BE + AF$; but $AB + DC = AB + FE$;

hence, Ax. 4, $AB + BC + DC + AD < AB + BE + FE + AF$;

that is, perimeter of $ABCD <$ perimeter of $ABEF$.

**Ex. 684.** Of all rectangles of a given area, the square has the least perimeter.



**Data**: Square $A$ and any other rectangle, $B$, equivalent to $A$.

**To prove** perimeter of $A <$ perimeter of $B$.

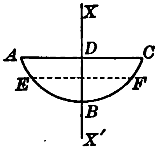
**Proof.** Construct the square $C$ with perimeter equal to that of $B$.

Then, § 414, $C > B$; but, data, $A \simeq B$;

$$\therefore\ C > A.$$

Since $A$ and $C$ are both squares, the perimeter of $C$ must be greater than that of $A$; consequently, the perimeter of $A$ must be less than that of $B$, which is equal to the perimeter of $C$.

**Ex. 685.** A segment of a circle is symmetrical with respect to the perpendicular bisector of its chord as an axis.



**Data**: Segment $ABC$ and the perpendicular bisector of its chord, $XX'$, cutting the arc at $B$.

**To prove** $ABC$ symmetrical with respect to $XX'$.

**Proof.** From any point in the arc $AB$, as $E$, draw $EF \perp XX'$, meeting arc $BC$ as in $F$.

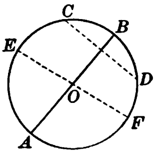
§ 199, $XX'$ passes through the center of the circle of which the given segment is a part;

$\therefore$ § 200, $XX'$ bisects $EF$;

that is, § 420, $E$ and $F$ are symmetrical with respect to $XX'$.

But $E$ is any point in the arc $AB$; $\therefore$ every point in arc $AB$ has a symmetrical point in arc $BC$, and, § 421, $ABD$ is symmetrical to $BCD$; that is, § 423, $ABC$ is symmetrical with respect to $XX'$.

**Ex. 686.** A circle is symmetrical with respect to its center or with respect to any diameter as an axis.



**Data**: A circle whose center is $O$ and any diameter, as $AB$.

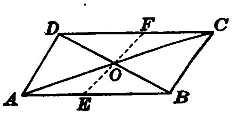
**To prove** the circle symmetrical with respect to $O$ and $AB$.

**Proof.** 1. Through $O$ draw any line, as $EF$, terminated in the circumference.

Then, Ax. 14, $OE = OF$; that is, $O$ bisects $EF$; but $EF$ is any line through $O$ terminated in the circumference; hence, § 422, the circle is symmetrical with respect to $O$.

2. § 192, $AB$ divides the circle into two equal parts which can be made to coincide by revolving one of them about $AB$ as an axis through 180°; $\therefore$ these semicircles are symmetrical with respect to $AB$ as an axis; that is, § 423, the circle is symmetrical with respect to $AB$.

**Ex. 687.** A parallelogram is symmetrical with respect to the point of intersection of its diagonals as a center.



**Data:** ▱ *ABCD* and *O* the point of intersection of its diagonals *AC* and *BD*.

**To prove** *ABCD* symmetrical with respect to *O*.

**Proof.** Through *O* draw any line, as *EF*, terminated in the boundary of *ABCD*.

By Ex. 133, *O* bisects *EF*;

but *EF* is any line through *O* terminated in the boundary of *ABCD*;

hence, § 422, *ABCD* is symmetrical with respect to *O*.

**Ex. 688.** If the perimeter of each of the figures, equilateral triangle, square, and circle, is 396 ft., what is the area of each figure?

**Solution.** 1. Side of $\triangle = 396 \div 3 = 132$ ft.

Then, Ex. 649, $r\sqrt{3} = 132$; whence, $r = \frac{132}{\sqrt{3}}$,

and apothem $= \frac{1}{2}r = \frac{66}{\sqrt{3}}$ ft.;

hence, § 389, area of $\triangle = \frac{1}{2} \times 396 \times \frac{66}{\sqrt{3}} = 7545$ sq. ft., nearly.

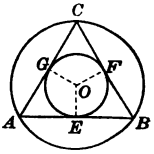
2. Side of square $= 396 \div 4 = 99$ ft.;

hence, area of square $= 99^2 = 9801$ sq. ft.

3. § 395, $C = 2\pi R = 396$; whence, $R = \frac{198}{\pi}$;

hence, § 398, area of $\odot = \pi R^2 = \pi \frac{198^2}{\pi^2} = \frac{39204}{3.1416} = 12478.99+$ sq. ft.

**Ex. 689.** If the inscribed and circumscribed circles of a triangle are concentric, the triangle is equilateral.



**Data:** $\triangle$ *ABC* whose inscribed and circumscribed circles are concentric with center at *O*.

**To prove** $\triangle$ *ABC* equilateral.

**Proof.** Draw *OE*, *OF*, and *OG* to the points of contact of the sides *AB*, *BC*, and *AC* of the $\triangle$ and the inscribed $\odot$.

Then, § 205, $OE \perp AB$, $OF \perp BC$, $OG \perp AC$;

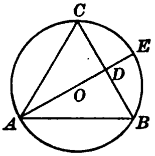
hence, § 62, *OE*, *OF*, and *OG* represent the distances from *O* to *AB*, *BC*, and *AC*, respectively;

but, Ax. 14, $OE = OF = OG$;

hence, § 202, $AB = BC = AC$;

that is, $\triangle$ *ABC* is equilateral.

**Ex. 690.** If an equilateral triangle is inscribed in a circle, any side will cut off one fourth of the diameter from the opposite vertex.



**Data:** Equilateral $\triangle ABC$ inscribed in the $\odot$ whose center is $O$; also the diameter $AE$ from $A$, cut at $D$ by $BC$.

**To prove** $DE = \frac{1}{4} AE$.

**Proof.** Since $A$ and $O$ are each equidistant from $B$ and $C$, § 106, $AE \perp BC$.

Then, $OD$ is the apothem of $\triangle ABC$,

and since, Ex. 649, $OD = \frac{1}{2} OE$,

$$DE = \tfrac{1}{2} OE = \tfrac{1}{4} AE.$$

**Ex. 691.** The square inscribed in a circle is equivalent to one half the square circumscribed about that circle.

**Proof.** The diagonal of the inscribed square = the diameter of the $\odot = 2R =$ the side of the circumscribed square, and the area of circumscribed square $= (2R)^2 = 4R^2$.

Denoting the side of the inscribed square by $x$,

§ 351, $\frac{2R}{x} = \sqrt{2}$; whence, $x = \frac{2R}{\sqrt{2}} = R\sqrt{2}$,

and the area of the inscribed square $= (R\sqrt{2})^2 = 2R^2$; consequently, the inscribed square $\approx \frac{1}{2}$ the circumscribed square.

**Ex. 692.** A circle is inscribed in a square whose side is 4 in. How much of the area of the square is without the circle?

**Solution.** The diameter of $\odot =$ side of square;

$\therefore$ $R = 2$ in., and, § 398, area of $\odot = \pi R^2 = 3.1416 \times 2^2 = 12.5664$ sq. in.

The area of the square $= 4^2 = 16$ sq. in.

Hence, area of the part of the square without the $\odot$

$$= 16 \text{ sq. in.} - 12.5664 \text{ sq. in.} = 3.4336 \text{ sq. in.}$$

**Ex. 693.** What is the width of the ring between the circumferences of two concentric circles whose circumferences are 48 ft. and 36 ft. respectively?

**Solution.** Denote the radii of the circles by $R$ and $R'$ respectively.

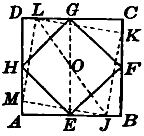
Then, § 395, $R = \frac{48}{2 \times 3.1416} = 7.639+$ ft.,

and, $R' = \frac{36}{2 \times 3.1416} = 5.729+$ ft.

Hence, width of ring $= R - R' = 7.639+$ ft. $- 5.729+$ ft. $= 1.91$ ft. nearly.

**Ex. 694.** Of all squares that can be inscribed in a given square, the minimum has its vertices at the middle points of the sides.

**Data:** Square $ABCD$, and the inscribed square $EFGH$ with its vertices at the middle points of the sides of $ABCD$.



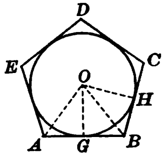
**To prove** $EFGH$ the minimum square that can be inscribed in $ABCD$.

**Proof.** Inscribe any other square, as $JKLM$, and draw its diagonal $LJ$, also draw the diagonal $GE$ of $EFGH$ intersecting $LJ$, as in $O$.

Now, Ax. 7, $EB = GC$, and, § 140, $EB \parallel GC$; $\therefore$ § 150, $EBCG$ is a ▱, and $EG \parallel BC$; but $AB \perp BC$; $\therefore$ § 72, $AB \perp EG$; that is, $EG \perp AB$ and $DC$; hence, § 61, $OE < OJ$, and $OG < OL$; consequently, $EG < LJ$, and $\overline{EG}^2 < \overline{LJ}^2$; but $\overline{EG}^2 = 2\,\overline{EF}^2$, and $\overline{LJ}^2 = 2\,\overline{MJ}^2$; hence, $2\,\overline{EF}^2 < 2\,\overline{MJ}^2$, or $\overline{EF}^2 < \overline{MJ}^2$; that is,

$$EFGH < JKLM.$$

**Ex. 695.** Every equiangular polygon circumscribed about a circle is regular.



**Data:** Any equiangular polygon, as $ABCDE$, circumscribed about the circle whose center is $O$.

**To prove** $ABCDE$ a regular polygon.

**Proof.** Draw $OA$, $OB$, and draw $OG$, $OH$ to the points of contact of $AB$ and $BC$ respectively.

Then, in rt. $\triangle$s $AGO$ and $BGO$, $GO$ is common;
since, Ex. 221, $OA$ and $OB$ bisect $\angle$s $A$ and $B$, Ax. 7, $\angle OAG = \angle OBG$;
$\therefore$ § 114, $\triangle AGO = \triangle BGO$, and $AG = BG$; similarly, $BH = HC$;
that is, $BG = \frac{1}{2} AB$, and $BH = \frac{1}{2} BC$;
but, § 209, $BG = BH$; consequently, $AB = BC$.
Similarly, $BC = CD =$ etc.
Hence, § 374, $ABCDE$ is a regular polygon.

**Ex. 696.** In any regular polygon of an even number of sides, the lines joining opposite vertices are diameters of the circumscribing circle.

**Proof.** Whatever the number of sides (if even) any chord joining opposite vertices subtends one half the whole number of equal arcs subtended by the sides of the polygon, or one half the circumference of the circle.

Then, this chord is a diameter.

**Ex. 699.** The sides of an inscribed rectangle are $30^{\text{cm}}$ and $40^{\text{cm}}$. What is the area of the part of the circle without the rectangle?

**Solution.** Let $ABCD$ be an inscribed rectangle whose side $AB = 30^{\text{cm}}$ and side $BC = 40^{\text{cm}}$.

Diag. $AC = \sqrt{\overline{AB}^2 + \overline{BC}^2} = \sqrt{30^2 + 40^2} = 50^{\text{cm}}$, and since $AC$ is a diameter of the $\odot$, the radius $= \frac{1}{2}$ of $50^{\text{cm}} = 25^{\text{cm}}$;
hence, § 398, area of $\odot = \pi R^2 = 3.1416 \times 25^2 = 1963.5^{\text{sq cm}}$.
Area of $ABCD = AB \times BC = 30 \times 40 = 1200^{\text{sq cm}}$.
Hence,
area of part of $\odot$ outside of $ABCD = 1963.5^{\text{sq cm}} - 1200^{\text{sq cm}} = 763.5^{\text{sq cm}}$.

**Ex. 700.** What is the area of a figure bounded by four semicircumferences described on the sides of a three-foot square?

**Solution.** Let the diameter of each semicircle $= D = 3$ ft.

§ 398, area of each semicircle $= \frac{1}{2}\pi R^2 = \frac{1}{2}\pi\left(\frac{D}{2}\right)^2 = \frac{1}{8}\pi D^2$;

$\therefore$ area of the 4 semicircles $= 4 \times \frac{1}{8}\pi D^2 = \frac{1}{2} \times 3.1416 \times 3^2 = 14.1372$ sq. ft.

Area of the square $= 3^2 = 9$ sq. ft.;

hence, the total area of the figure $= 14.1372$ sq. ft. $+ 9$ sq. ft. $= 23.1372$ sq. ft.

**Ex. 701.** A square piece of land and a circular piece of land each contain one acre. Which perimeter is the greater, and how much?

**Solution.** 1 acre $= 160$ sq. rd.

Then, side of square $= \sqrt{160} = 12.649+$ rd.,

and perimeter $= 4 \times 12.649+$ rd. $= 50.596+$ rd.;

§ 398, area of circle $= \pi R^2 = 160$; hence, $R = \sqrt{\frac{160}{\pi}}$.

Then, § 395, $C = 2\pi R = 2\pi\frac{\sqrt{160}}{\sqrt{\pi}} = 2\sqrt{\pi} \times \sqrt{160} = 44.84-$ rd.

Hence, the perimeter of the square is greater than the perimeter of the circle by $50.596+$ rd. $- 44.84-$ rd. $= 5.756+$ rd.

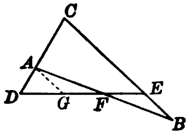
**Ex. 702.** The area of an inscribed equilateral triangle is one half the area of a regular hexagon inscribed in the same circle.

**Proof.** Ex. 649, § 389, area of $\triangle = \frac{1}{2} \times 3r\sqrt{3} \times \frac{1}{2}r = \frac{3}{4}r^2\sqrt{3}$.

Ex. 648, § 389, area of hexagon $= \frac{1}{2} \times 6r \times \frac{1}{2}r\sqrt{3} = \frac{3}{2}r^2\sqrt{3}$.

Hence, area of $\triangle = \frac{1}{2}$ area of hexagon.

**Ex. 703.** Of all triangles that have the same vertical angle and whose bases pass through a given point, the minimum is the one whose base is bisected at that point.



**Data:** Any two $\triangle$s, as $ABC$ and $DEC$, having the same vert. $\angle C$ and their bases passing through point $F$, $AB$ being bisected at $F$.

**To prove** $\triangle ABC$ the minimum.

**Proof.** Suppose that $DF$ is the longer segment of $DE$. Take $FG = FE$, and draw $AG$.

Then, since, § 59, $\angle BFE = \angle AFG$, § 100, $\triangle BEF = \triangle AGF$;

consequently, $\triangle BEF < \triangle AFD$;

hence, adding $AFEC$ to both members of this inequality,

$$AFEC + \triangle BEF < \triangle AFD + AFEC,$$

or $\triangle ABC < \triangle DEC$;

that is, $\triangle ABC$ is the minimum.

**Ex. 704.** An arc of a circle whose radius is 6 ft. subtends a central angle of 20°; an equal arc of another circle subtends a central angle of 30°. What is the radius of the second circle?

**Solution.** Denote the circumferences and radii of the 1st and 2d Ⓢ by $C$ and $C'$, and $R$ and $R'$ resp., and the given arc by $a$.

Then, in the 1st ⊙, $a = \frac{20}{360} C = \frac{1}{18} C$; whence, $C = 18a$.

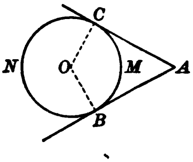
and in the 2d ⊙, $a = \frac{30}{360} C' = \frac{1}{12} C'$; whence, $C' = 12a$;

hence, $C : C' = 18a : 12a$; but, § 394, $C : C' = R : R'$;

∴ $R : R' = 18a : 12a$; that is, $6 : R' = 18a : 12a$;

whence $$R' = \frac{6 \times 12a}{18a} = 4 \text{ ft.}$$

**Ex. 705.** Two tangents make with each other an angle of 60°, and the radius of the circle is 7 in. What are the lengths of the arcs between the points of contact?



**Solution.** Let $AB$ and $AC$ be tangents to the ⊙ whose center is $O$ and radius 7 in., $\angle BAC$ being 60°.

Draw the radii $OB$ and $OC$.

Then, in the quadrilateral $ABOC$, since, § 205, △ $ABO$ and $ACO$ are rt. △, $\angle BOC$ is the sup. of $\angle BAC$;

hence, $\angle BOC = 180° - 60° = 120°$;

consequently, arc $BMC = 120° = \frac{120}{360}$, or $\frac{1}{3}$ of the circumference.

§ 395, circum. $= 2\pi R = 2 \times 3.1416 \times 7 = 43.9824$ in.;

hence, arc $BMC = \frac{1}{3}$ of 43.9824 in. $=$ 14.6608 in.,

and arc $BNC = \frac{2}{3}$ of 43.9824 in. $=$ 29.3216 in.

**Ex. 706.** If the apothem of a regular hexagon is $10^{\text{m}}$, what is the area of the ring between the circumferences of its inscribed and circumscribed circles?

**Solution.** Ex. 648, data, apothem $= \frac{1}{2} r\sqrt{3} = 10$;

whence, $$r = \frac{20}{\sqrt{3}}.$$

Hence, § 395, $$\text{area of ring} = \pi\left(\frac{20}{\sqrt{3}}\right)^2 - \pi\, 10^2 = \pi\left(\frac{400}{3} - 100\right)$$
$$= \pi\left(\frac{400 - 300}{3}\right) = \frac{314.16}{3} = 104.72^{\text{m}}.$$

**Ex. 707.** If a circle $18^{\text{cm}}$ in diameter is divided into three equivalent parts by two concentric circumferences, what are their radii?

**Solution.** Denote the required radii by $r$ and $r'$, $r$ being the larger. Denote the area of the given ⊙ by 1; then, that of the circle whose radius is $r$ will be $\frac{2}{3}$ and that of the circle whose radius is $r'$ will be $\frac{1}{3}$.

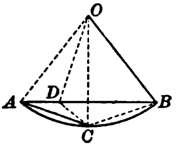
§ 399, $1 : \frac{2}{3} = 9^2 : r^2$; whence, $r^2 = \frac{2}{3} \times 9^2 = 54$,

and $r = \sqrt{54} = 7.348+^{\text{cm}}$.

Similarly, $1 : \frac{1}{3} = 9^2 : r'^2$; whence, $r'^2 = \frac{1}{3} \times 9^2 = 27$,

and $r' = \sqrt{27} = 5.196+^{\text{cm}}$.

**Ex. 708.** The square upon the side of a regular inscribed pentagon is equivalent to the sum of the squares upon the radius of the circle and the side of a regular inscribed decagon.



**Data:** The side $AB$ of a regular inscribed pentagon, the side $AC$ of a regular inscribed decagon, and $OB$ the radius of the circumscribing circle.

**To prove** $\overline{AB}^2 \Bumpeq \overline{AC}^2 + \overline{OB}^2$.

**Proof.** Draw $OA$, $OC$, and draw $OD$, bisecting $\angle AOC$ and meeting $AB$ in $D$; also draw $DC$ and $BC$.

Then, in $\triangle$s $OAD$ and $OCD$,

$$OA = OC,\ OD \text{ is common, and } \angle AOD = \angle COD;$$

∴ § 100, $\triangle OAD = \triangle OCD$, and $AD = DC$;

hence, § 90, $\triangle ADC$ is isosceles;

but $\triangle ACB$ is also isosceles;

∴ § 301, $\triangle$s $ADC$ and $ACB$ are similar.

Hence, § 299, $AB : AC = AC : AD$;

whence, § 269, $AB \times AD = \overline{AC}^2$. (1)

Now, $\angle AOB = \frac{1}{5}$ of 360°, or 72°,

$\angle AOC = \frac{1}{10}$ of 360°, or 36°.

and, const., $\angle COD = \frac{1}{2}\angle AOC = 18°$;

then, $\angle DOB = \angle COD + \angle BOC = 18° + 36° = 54°$.

But, in isosceles $\triangle ABO$,

$$\angle ABO = \tfrac{1}{2}(180° - \angle AOB) = 54°;$$

consequently, § 118, $\triangle DBO$ is isosceles,

and since $\triangle ABO$ is isosceles,

§ 301, $\triangle$s $ABO$ and $DBO$ are similar;

hence, $AB : OB = OB : DB$;

whence, $AB \times DB = \overline{OB}^2$. (2)

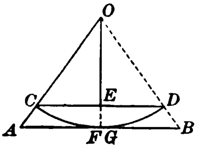
Adding (1) and (2),

$$AB \times AD + AB \times DB = \overline{AC}^2 + \overline{OB}^2;$$

but $AB \times AD + AB \times DB = AB(AD + DB) = AB \times AB = \overline{AB}^2$;

hence, $\overline{AB}^2 \Bumpeq \overline{AC}^2 + \overline{OB}^2$.

**Ex. 709.** The radius of a regular inscribed polygon is a mean proportional between its apothem and the radius of the similar circumscribed polygon.



**Data:** $CD$, the side of a regular inscribed polygon whose radius is $OC$ and apothem $OE$, and $AB$ the side of a similar circumscribed polygon whose radius is $OA$.

**To prove** $OE : OC = OC : OA$.

**Proof.** Place $AB \parallel CD$; then, § 385, $O$, $C$, and $A$ are in the same straight line.

Produce $OE$ to meet $AB$, as at $F$; draw a line from $O$ to $G$, the point of contact of $AB$.

Then, § 205, $OG \perp AB$, but since $OEF \perp CD$, § 72, $OEF \perp AB$; consequently, § 60, $OF$ and $OG$ coincide, and $F$ and $G$ coincide, being the point of contact.

Now, in rt. △ $CEO$ and $AFO$, $\angle O$ is common;

∴ § 302, △ $CEO$ and $AFO$ are similar;

hence, $OE : OC = OF : OA$;

but, Ax. 14, $OF = OC$;

consequently, $OE : OC = OC : OA$.

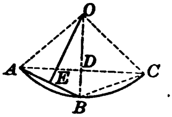
**Another Proof.** Let $h$ = the apothem of the inscribed polygon, $r$ its radius, $R$ the radius of the circumscribed polygon, and $p$ and $P$ the perimeters of the inscribed and the circumscribed polygon, respectively.

Then, § 387, $p : P = h : r$, and $p : P = r : R$;

hence, $h : r = r : R$.

Ex. 710. If the radius of a regular inscribed octagon is $r$, prove that its side $= r\sqrt{2-\sqrt{2}}$, and its apothem $= \frac{r}{2}\sqrt{2+\sqrt{2}}$.

**Data:** $AB$ the side of a regular inscribed octagon whose radius is $r$ and apothem $OE$.



**To prove** $AB = r\sqrt{2-\sqrt{2}}$, and $OE = \frac{r}{2}\sqrt{2+\sqrt{2}}$.

**Proof.** Draw $AC$ the side of an inscribed square; draw the radii $OA$, $OC$, and $OB$, $OB$ cutting $AC$ as at $D$; and draw $BC$.

Since arc $ABC = \frac{1}{4}$ of circumference, and arc $AB = \frac{1}{8}$ of circumference, arc $BC = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$ of circumference; consequently, $BC$ is the side of a regular inscribed octagon, and $AB = BC$, also $OA = OC$; consequently, § 106, $OB \perp AC$ at its middle point $D$.

Ex. 650, $OD = \frac{1}{2}r\sqrt{2}$; $BD = OB - OD = r - \frac{1}{2}r\sqrt{2} = \frac{r}{2}(2-\sqrt{2})$;

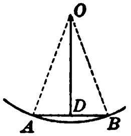
also, $AC = r\sqrt{2}$; consequently, $AD = \frac{1}{2}r\sqrt{2}$;

then, § 349, $\overline{AB}^2 = \overline{AD}^2 + \overline{BD}^2 = (\frac{1}{2}r\sqrt{2})^2 + \left[\frac{r}{2}(2-\sqrt{2})\right]^2 = r^2(2-\sqrt{2})$,

and $AB = \sqrt{r^2(2-\sqrt{2})} = r\sqrt{2-\sqrt{2}}$.

§ 350, $OE = \sqrt{\overline{OA}^2 - \overline{AE}^2} = \sqrt{r^2 - (\frac{1}{2}r\sqrt{2-\sqrt{2}})^2} = \frac{r}{2}\sqrt{2+\sqrt{2}}$.

**Ex. 711.** If the radius of a regular inscribed decagon is $r$, prove that its side $= \frac{r}{2}(\sqrt{5} - 1)$, and its apothem $= \frac{r}{4}\sqrt{10 + 2\sqrt{5}}$.



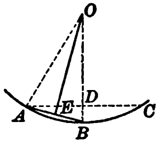
**Data:** $AB$ the side of a regular inscribed decagon whose radius is $r$ and apothem $OD$.

**To prove** $AB = \frac{r}{2}(\sqrt{5} - 1)$, and $OD = \frac{r}{4}\sqrt{10 + 2\sqrt{5}}$.

**Proof.** § 405, $OA : AB = AB : OA - AB$;
that is, $r : AB = AB : r - AB$; whence, $\overline{AB}^2 = r^2 - ABr$;
solving, $AB = \frac{r}{2}(\sqrt{5} - 1)$.

§ 350, $OD = \sqrt{\overline{OA}^2 - \overline{AD}^2} = \sqrt{r^2 - \left[\frac{1}{2} \times \frac{r}{2}(\sqrt{5} - 1)\right]^2} = \frac{r}{4}\sqrt{10 + 2\sqrt{5}}$.

**Ex. 712.** If the radius of a regular inscribed dodecagon is $r$, prove that its side $= r\sqrt{2 - \sqrt{3}}$, and its apothem $= \frac{r}{2}\sqrt{2 + \sqrt{3}}$.



**Data:** $AB$ the side of a regular inscribed dodecagon whose radius is $r$ and apothem $OE$.

**To prove** $AB = r\sqrt{2 - \sqrt{3}}$, and $OE = \frac{r}{2}\sqrt{2 + \sqrt{3}}$.

**Proof.** Draw $AC$ the side of a regular inscribed hexagon, and proceeding as in Ex. 710, it may be shown that $OB \perp AC$ at its middle point $D$.

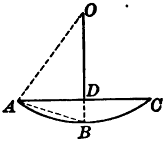
Then, Ex. 648, $AD = \frac{1}{2} AC = \frac{1}{2} r$,
and $BD = OB - OD = r - \frac{1}{2} r\sqrt{3} = \frac{r}{2}(2 - \sqrt{3})$.

Now, § 349, $\overline{AB}^2 = \overline{AD}^2 + \overline{BD}^2 = (\frac{1}{2} r)^2 + \frac{r^2}{4}(2 - \sqrt{3})^2 = r^2(2 - \sqrt{3})$;

whence, $AB = r\sqrt{2 - \sqrt{3}}$.

§ 350, $OE = \sqrt{\overline{OA}^2 - \overline{AE}^2} = \sqrt{r^2 - (\frac{1}{2} r\sqrt{2 - \sqrt{3}})^2} = \frac{r}{2}\sqrt{2 + \sqrt{3}}$.

**Ex. 713.** If the radius of a regular inscribed pentagon is $r$, prove that its side $= \frac{r}{2}\sqrt{10 - 2\sqrt{5}}$, and its apothem $= \frac{r}{4}\sqrt{6 + 2\sqrt{5}}$.



**Data:** $AC$ the side of a regular inscribed pentagon, whose radius is $r$ and apothem $OD$.

**To prove** $AC = \frac{r}{2}\sqrt{10 - 2\sqrt{5}}$, and $OD = \frac{r}{4}\sqrt{6 + 2\sqrt{5}}$.

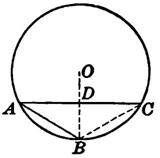
**Proof.** Draw $AB$, the side of a regular inscribed decagon, and draw $OA$.

Then, Ex. 708, $\overline{AC}^2 = r^2 + \overline{AB}^2$; but, Ex. 711, $AB = \frac{r}{2}(\sqrt{5} - 1)$;

hence, $\overline{AC}^2 = r^2 + \left(\frac{r}{2}(\sqrt{5} - 1)\right)^2 = \frac{r^2}{4}(10 - 2\sqrt{5})$, and $AC = \frac{r}{2}\sqrt{10 - 2\sqrt{5}}$.

§ 350, $OD = \sqrt{\overline{OA}^2 - \overline{AD}^2} = \sqrt{r^2 - \left(\frac{1}{2} \times \frac{r}{2}\sqrt{10 - 2\sqrt{5}}\right)^2} = \frac{r}{4}\sqrt{6 + 2\sqrt{5}}$

Ex. **714.** The square upon a side of an inscribed equilateral triangle is equivalent to three times the square upon the side of a regular inscribed hexagon.



**Data:** $AC$ the side of an inscribed equilateral $\triangle$, and $AB$ the side of a regular inscribed hexagon.

**To prove** $\overline{AC}^2 \simeq 3\,\overline{AB}^2$.

**Proof.** Draw $BC$ and draw the radius $OB$ cutting $AC$ as at $D$.

As in Ex. 710, it may be shown that $BC = AB$ and $OB \perp AC$ at its middle point $D$.

§ 349, $\overline{AD}^2 \simeq \overline{AB}^2 - \overline{BD}^2$, and $\overline{DC}^2 \simeq \overline{BC}^2 - \overline{BD}^2$,

adding, $$\overline{AD}^2 + \overline{DC}^2 \simeq \overline{AB}^2 + \overline{BC}^2 - 2\,\overline{BD}^2, \quad (1)$$

But $AD = \frac{1}{2}AC$, $DC = \frac{1}{2}AC$, $BC = AB$, and, Ex. 649, 648, $BD = \frac{1}{2}AB$.

Substituting in (1), $\frac{1}{4}\overline{AC}^2 + \frac{1}{4}\overline{AC}^2 \simeq \overline{AB}^2 + \overline{AB}^2 - \frac{1}{2}\overline{AB}^2$;

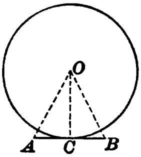
whence, $\overline{AC}^2 \simeq 3\,\overline{AB}^2$.

Ex. **715.** The area of an inscribed square is $16^{\text{sq m}}$. Find the length of a side of a regular inscribed octagon.

**Solution.** Ex. 650, data, $r\sqrt{2} = \sqrt{16}$; whence, $r = 2\sqrt{2}$.
Then, Ex. 710, side of regular inscribed octagon

$$= r\sqrt{2-\sqrt{2}} = 2\sqrt{2}\sqrt{2-\sqrt{2}} = 2\sqrt{4-2\sqrt{2}}.$$

Ex. **716.** If the radius of a circle is $r$, prove that a side of a regular circumscribed hexagon is $\frac{2r}{3}\sqrt{3}$.



**Data:** $AB$ the side of a regular hexagon circumscribed about a circle whose center is $O$ and radius $r$.

**To prove** $AB = \frac{2r}{3}\sqrt{3}$.

**Proof.** Draw $OC$ to the point of contact of $AB$, and draw $OA$ and $OB$.

§ 381 and Ex. 648, $\triangle ABO$ is equilateral, and since, § 205, $OC \perp AB$,

§ 122, $AC = \frac{1}{2}AB$.

§ 350, $AC = \sqrt{\overline{AO}^2 - \overline{OC}^2}$; that is, $\frac{AB}{2} = \sqrt{\overline{AB}^2 - r^2}$;

hence, $AB = 2\sqrt{\overline{AB}^2 - r^2}$, and $\overline{AB}^2 = 4(\overline{AB}^2 - r^2) = 4\,\overline{AB}^2 - 4r^2$;

whence, $\overline{AB}^2 = \frac{4r^2}{3}$, and $AB = \frac{2r}{3}\sqrt{3}$.

**Ex. 717.** The area of a regular inscribed dodecagon is equal to three times the square of the radius.

**Proof.** Ex. 712, side $= r\sqrt{2-\sqrt{3}}$; consequently, the perimeter

$$= 12\,r\sqrt{2-\sqrt{3}}, \text{ and the apothem } = \frac{r}{2}\sqrt{2+\sqrt{3}};$$

hence, § 389,

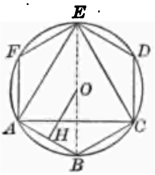
the area $= \frac{1}{2} \times 12\,r\sqrt{2-\sqrt{3}} \times \frac{r}{2}\sqrt{2+\sqrt{3}} = 3\,r^2\sqrt{(2-\sqrt{3})(2+\sqrt{3})} = 3\,r^2.$

**Ex. 718.** Find the side of a regular hexagon circumscribed about a circle whose diameter is 1.

**Solution.** Ex. 716, side of regular circumscribed hexagon $= \frac{2\,r}{3}\sqrt{3}$; but since the diameter of the ⊙ is 1, $r = \frac{1}{2}$;

∴ substituting, the required side $= \frac{2(\frac{1}{2})}{3}\sqrt{3} = \frac{1}{3}\sqrt{3}.$

**Ex. 719.** The apothem of an inscribed regular hexagon is equal to one half the side of the inscribed equilateral triangle.



**Data:** Regular inscribed hexagon $ABCDEF$; its apothem $OH$; and $EA$ a side of the inscribed equilateral $\triangle ACE$.

**To prove** $OH = \frac{1}{2}EA$.

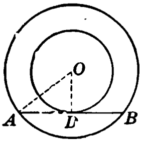
**Proof.** Draw $EB$; then, Ex. 696, $EB$ is a diameter of the circumscribing ⊙ and passes through the center $O$;

∴ § 227, $\angle EAB$ is a rt. ∠; also $\angle OHB$ is a rt. ∠;

hence, § 71, $OH \parallel EA$, and since, Ax. 14, $OE = OB$,

§ 158, $OH = \frac{1}{2}EA.$

**Ex. 720.** The area of a ring bounded by two concentric circumferences is equal to the area of a circle whose diameter is a chord of the outer circumference and is tangent to the inner circumference.



**Data:** Two concentric circumferences whose center is $O$, also chord $AB$ of the outer circumference tangent to the inner circumference at $D$.

**To prove** area of ring bounded by given circumferences equal to the area of a circle whose diameter is $AB$.

**Proof.** Draw the radii $OA$ and $OD$.

§ 398, area of outer ⊙ $= \pi\overline{OA}^2$, and area of inner ⊙ $= \pi\overline{OD}^2$;

∴ area of ring $= \pi\overline{OA}^2 - \pi\overline{OD}^2 = \pi(\overline{OA}^2 - \overline{OD}^2)$;

but since, § 205, $OD \perp AB$, $ADO$ is a rt. $\triangle$, and $\overline{OA}^2 - \overline{OD}^2 = \overline{AD}^2$;

∴ area of ring $= \pi\overline{AD}^2$, and since, § 200, $AD = \frac{1}{2}AB$,

area of ring = area of a circle whose diameter is $AB$.

Ex. **721.** If the radius of a circle is $r$, find the area of a segment whose chord is one side of a regular inscribed hexagon.

**Solution.** The arc of the segment $= \frac{1}{6} \times 2\pi r = \frac{1}{3}\pi r$;

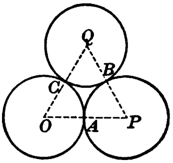
$\therefore$ the area of the corresponding sector $= \frac{1}{2} \times \frac{1}{3}\pi r \times r = \frac{1}{6}\pi r^2$.

Ex. 648, the altitude of the corresponding $\Delta$ is $\frac{1}{2} r\sqrt{3}$ and its base is $r$;

$\therefore$ § 335, the area of the corresponding $\Delta = \frac{1}{2} \times r \times \frac{1}{2} r\sqrt{3} = \frac{1}{4} r^2\sqrt{3}$.

Hence, area of segment $= \frac{1}{6}\pi r^2 - \frac{1}{4} r^2\sqrt{3} = \frac{r^2}{12}(2\pi - 3\sqrt{3})$.

Ex. **722.** Three equal circles with a radius of 12 ft. are drawn tangent to each other. What is the area between them?



**Solution.** Let the equal Ⓢ whose centers are $O$, $P$, and $Q$ be tangent to each other at the points $A$, $B$, and $C$.

Draw $OP$, $PQ$, and $OQ$; then, § 213, these lines pass through $A$, $B$, and $C$ respectively, and the area $ABC$ (between the Ⓢ) = area $\Delta\, OPQ$ − area (sect. $OAC$ + sect. $PAB$ + sect. $QCB$)

$$= \text{area } \Delta\, OPQ - 3 \text{ area sect. } OAC. \quad (1)$$

Since a side of $\Delta\, OPQ$ = twice the radius = 12 ft. × 2 = 24 ft.,

Ex. 632, area $\Delta\, OPQ = \sqrt{36 \times 12 \times 12 \times 12} = 144\sqrt{3} = 249.4152+$ sq. ft.

Since $\angle O = 60°$, arc $AC = \frac{60}{360}$, or $\frac{1}{6}$ of the circumference $= \frac{1}{6} \times 2\pi \cdot 12 = 4 \times 3.1416 = 12.5664$;

consequently, § 400, area sect. $OAC = \frac{1}{2} \times 12 \times 12.5664 = 75.3984$ sq. ft.,

and 3 area sect. $OAC = 3 \times 75.3984$ sq. ft. $= 226.1952$ sq. ft.

Substituting in (1),

area $ABC = 249.4152+$ sq. ft. $- 226.1952$ sq. ft. $= 23.22+$ sq. ft.

Ex. **723.** The area of an inscribed regular hexagon is equal to three fourths that of a regular hexagon circumscribed about the same circle.

**Proof.** Ex. 648, side of inscribed hexagon $= r$,

and, Ex. 716, side of circumscribed hexagon $= \frac{2r}{3}\sqrt{3}$.

§ 386, these polygons are similar;

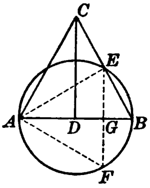
$\therefore$ § 344, inscribed hexagon : circumscribed hexagon $= r^2 : \left(\frac{2r}{3}\sqrt{3}\right)^2 =$

$$r^2 : \frac{4r^2}{3} = 3r^2 : 4r^2 = 3 : 4;$$

that is, area of inscribed hexagon $= \frac{3}{4}$ area of circumscribed hexagon.

Ex. **724.** The altitude of an equilateral triangle is equal to the side of an equilateral triangle inscribed in a circle whose diameter is the base of the first triangle.

**Data**: Equilateral $\triangle ABC$; its altitude $CD$; and the $\odot$ whose diameter is $AB$, the circumference cutting $BC$ in $E$.



**To prove** $CD$ equal to the side of an equilateral $\triangle$ inscribed in the $\odot$.

**Proof.** Draw $AE$; draw $EF \parallel CD$ cutting $AB$ as in $G$ and meeting the circumference as in $F$; and draw $AF$.

§ 227, $\angle AEB$ is a rt. $\angle$; $\therefore$ § 122, $CE = EB$;

hence, § 158, $EG = \frac{1}{2}CD$; but, § 200, $EG = \frac{1}{2}EF$;

consequently, $\frac{1}{2}CD = \frac{1}{2}EF$; that is, $CD = EF$;

Since $\triangle ABC$ is equilateral, § 94, $CD = AE$; also, § 103, $AF = AE$;

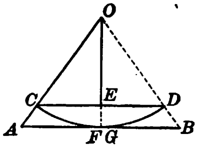
hence, $AF = AE = EF = CD$;

that is, $\triangle AFE$ is an inscribed equilateral $\triangle$,

and $CD =$ the side of $\triangle AFE$.

**Ex. 725.** If the radius of a circle is $r$, and the side of a regular inscribed polygon is $a$, prove that the side of a similar circumscribed polygon is

$$\frac{2ar}{\sqrt{4r^2 - a^2}}.$$



**Data**: $CD = a$, the side of a regular polygon inscribed in a $\odot$ whose radius is $r$, and $AB$ the side of a circumscribed polygon.

**To prove** $AB = \frac{2ar}{\sqrt{4r^2 - a^2}}$.

**Proof.** Make the same construction as in Ex. 709, and in the same way show that $F$ is a point of the circumference.

§ 76, $\angle OCD = \angle OAB$, and $\angle ODC = \angle OBA$;

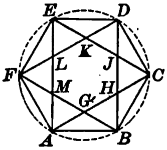
hence, § 301, $\triangle$s $CDO$ and $ABO$ are similar;

$\therefore OE : OF = CD : AB$; but $OE = \sqrt{\overline{OC}^2 - \overline{CE}^2} = \sqrt{r^2 - \left(\frac{a}{2}\right)^2} = \frac{1}{2}\sqrt{4r^2 - a^2}$.

Consequently, substituting, $\frac{1}{2}\sqrt{4r^2 - a^2} : r = a : AB$;

whence, $AB = \frac{ar}{\frac{1}{2}\sqrt{4r^2 - a^2}} = \frac{2ar}{\sqrt{4r^2 - a^2}}$.

**Ex. 726.** If the alternate vertices of a regular hexagon are joined by straight lines, another regular hexagon is formed which is one third as large as the original hexagon.



**Data**: Regular hexagon $ABCDEF$ and the lines joining its alternate vertices and intersecting each other in $G$, $H$, $J$, etc.

**To prove** $GHJKLM$ a regular hexagon, one third as large as $ABCDEF$.

**Proof.** 1. Circumscribe a circle about $ABCDEF$.

Then, § 225, $\angle GBH = 60°$, being measured by $\frac{1}{2}$ arc $FED$, or by $\frac{1}{6}$ of a circumference,

§ 230, $\angle BGH = 60°$, being measured by $\frac{1}{2}$ (arc $BC$ + arc $AF$), or by $\frac{1}{6}$ of a circumference,

and $\angle BHG = 60°$, being measured by $\frac{1}{2}$ (arc $AB$ + arc $CD$), or by $\frac{1}{6}$ of a circumference;

hence, § 119, $\triangle BHG$ is equilateral; similarly, $\triangle HCJ$ is equilateral.

Now, since arc $AB$ = arc $CD$, § 225, $\angle BCH = \angle CBH$,

and, § 118, $BH = CH$; $\therefore$ § 107, $\triangle BHG = \triangle HCJ$.

Similarly, $\triangle$s $HCJ$, $JDK$, $LKE$, etc., are equal and equilateral;

hence, $GH = HJ = JK =$ etc.; that is, $GHJKLM$ is equilateral,

and since its $\angle$s are the supplements of the $\angle$s of equiangular $\triangle$s,

§ 54, $GHJKLM$ is equiangular;

consequently, § 374, $GHJKLM$ is a regular hexagon.

2. § 108,

$AG = GH = HC$; $\therefore GH = \frac{1}{3} AC$, the side of an inscribed equilateral $\triangle$;

hence, Ex. 649, each side of $GHJKLM = \frac{1}{3} r\sqrt{3}$;

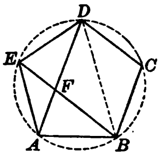
also, Ex. 648, each side of $ABCDEF = r$.

Now, § 386, $ABCDEF$ and $GHJKLM$ are similar;

hence, § 344, $ABCDEF : GHJKLM = r^2 : (\frac{1}{3} r\sqrt{3})^2 = r^2 : \frac{r^2}{3} = 3r^2 : r^2 = 3 : 1$;

that is, $GHJKLM$ is one third as large as $ABCDEF$.

Ex. **727**. The diagonals of a regular pentagon divide each other in extreme and mean ratio.



**Data**: Regular pentagon $ABCDE$ and any two of its diagonals, as $AD$ and $BE$, intersecting as at $F$.

**To prove** $AD$ and $BE$ divide each other in extreme and mean ratio.

**Proof.** Circumscribe a circle about $ABCDE$ and draw the diagonal $BD$.

Since arc $BCD$ = arc $AED$, § 225, $\angle DAB = \angle ABD$, and, § 118, $\triangle ABD$ is isosceles.

Since arc $BCD$ = arc $AB$ + arc $ED$, §§ 225, 230, $\angle DAB = \angle AFB$, and $\triangle ABF$ is isosceles.

Hence, § 301, $\triangle$s $ABD$ and $ABF$ are similar;

and, § 299, $AD : BF = AB : AF$. (1)

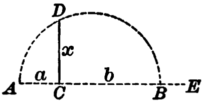
Now, since arc $ED$ = arc $AB$, § 225, $\angle FBD = \angle FDB$;

$\therefore$ § 118, $DF = BF = AB$.

Substituting in (1) $AD : DF = DF : AF$;
that is, $AD$ is divided in extreme and mean ratio at $F$.

Similarly, from △ $EBD$ and $EFD$ it may be shown that $BE$ is divided in extreme and mean ratio at $F$.

**Ex. 728.** Construct $x$, if $x = \sqrt{ab}$.



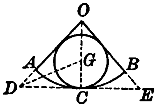
**Solution.** Draw the indefinite line $AE$ and on it take $AC = a$ and $CB = b$.

On $AB$ as a diameter describe a semicircumference and at $C$ erect a perpendicular to $AB$ meeting the semicircumference as at $D$.

Then, $CD$ is $x$, the required line.

**Proof.** § 314, $a : x = x : b$;
whence, § 269, $x^2 = ab$, and $x = \sqrt{ab}$.

**Ex. 729.** Inscribe a circle in a given sector.



**Solution.** Bisect $\angle O$ of the given sector $OAB$ and produce the bisector to meet the arc as at $C$.

Through $C$ draw a tangent to the arc, meeting $OA$ produced, as in $D$ and $OB$ produced, as in $E$.

Bisect $\angle ODE$ and produce the bisector to meet $OC$, as in $G$.

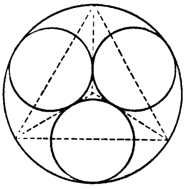
With center $G$ and radius $GC$ describe a ⊙.

Then, this ⊙ is the ⊙ required.

**Proof.** § 134, every point in $OC$ is equidistant from $OA$ and $OB$, and every point in $DG$ is equidistant from $OD$ and $DE$; ∴ $G$ the intersection of $OC$ and $DG$ is equidistant from $OA$, $OB$, and $DE$, and the ⊙ whose center is $G$ and radius $GC$ touches $OA$, $OB$, and $DE$; this ⊙ and the arc of the sector are tangent to $DE$ at the point $C$; ∴ § 185, the ⊙ and the arc are tangent to each other at $C$.

Hence, the ⊙ is inscribed in the sector.

**Ex. 730.** In a given circle describe three equal circles tangent to each other and to the given circle.



**Solution.** Divide the given ⊙ into three equal sectors by drawing radii to the vertices of an inscribed equilateral △.

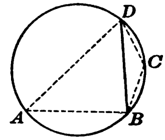
Ex. 729, inscribe Ⓢ in these sectors.

Then, these Ⓢ are the Ⓢ required.

**Proof.** Since the sectors are equal they can be made to coincide and their inscribed Ⓢ will coincide; hence the Ⓢ are equal and any two are tangent to the common radius of their sectors at the same point; ∴ § 185, the Ⓢ are tangent to each other, and since they are inscribed in the sectors they are tangent to the given ⊙.

**Ex. 731.** Divide a circle into two segments such that an angle inscribed in one shall be three times an angle inscribed in the other.

**Solution.** Since, § 225, each $\angle$ is measured by $\frac{1}{2}$ the arc of the segment containing the other, the sum of the measures of the two $\angle$s is $\frac{1}{2}$ the circumference, and the sum of the $\angle$s is 180°, and since one $\angle$ is three times the other, their sum is four times the smaller $\angle$, and the smaller $\angle = \frac{1}{4}$ of $180° = 45°$; consequently, at $A$, any point in the circumference of the given $\odot$, construct an $\angle$ of 45° and produce its sides to meet the circumference as at $B$ and $D$; draw $BD$.



Then, $BCD$ and $BAD$ are the required segments.

**Proof.** Construct any $\angle$, as $BCD$, in the segment $BCD$.

Then, Ex. 198, $\angle BCD + \angle BAD = 180°$; $\therefore$ $\angle BCD + 45° = 180°$; hence, $\angle BCD = 180° - 45° = 135°$; but $135° = 45° \times 3$, consequently, $\angle BCD = 3\angle BAD$, and the segments $BCD$ and $BAD$ are the required segments.

**Ex. 732.** Construct a circumference equal to the sum of two given circumferences.

**Solution.** Denote the given circumferences by $C$ and $C'$, and their radii by $R$ and $R'$.

With a radius $= R + R'$, describe the circumference $C''$.

Then, $C''$ is the required circumference.

**Proof.** § 395, $C = 2\pi R$, and $C' = 2\pi R'$;

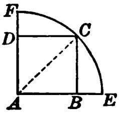
hence, $C + C' = 2\pi R + 2\pi R' = 2\pi(R + R')$;

but $C'' = 2\pi(R + R')$; hence, $C'' = C + C'$.

**Ex. 733.** Inscribe a square in a given quadrant.

**Solution.** Bisect the $\angle A$ of the given quadrant $AEF$ and produce the bisector to meet the arc $EF$ as at $C$.

From $C$ draw $CB \perp AE$, and $CD \perp AF$.



Then, $ABCD$ is the required square.

**Proof.** § 71, $AB \parallel DC$, and $AD \parallel BC$; $\therefore$ § 140, $ABCD$ is a $\square$, and since $\angle ABC$ is a rt. $\angle$, Ex. 104, $ABCD$ is a rectangle.

$\angle BAC = \frac{1}{2}$ rt. $\angle$; $\therefore$ § 111, $\angle ACB = \frac{1}{2}$ rt. $\angle$;

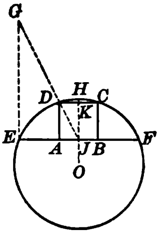
hence, §§ 118, 153, $AB = BC = CD = AD$;

consequently, $ABCD$ is a square.

**Ex. 734.** Inscribe a square in a given segment of a circle.

**Solution.** Draw the radius $OH \perp EF$, the chord of the given segment, cutting $EF$ as at $J$; draw $EG \perp EF$ and equal to $2\,EJ$; and draw $GJ$ cutting the arc as in $D$.

Draw $DA \perp EF$, and $DC \parallel EF$ cutting $OH$ as in $K$ and the arc as in $C$.



Draw $CB \perp EF$.

Then, $ABCD$ is the required square.

**Proof.** § 71, $AD \parallel BC$, and const., $AB \parallel DC$; $\therefore$ § 140, $ABCD$ is a ▱, and since $\angle ABC$ is a rt. $\angle$, Ex. 104, $ABCD$ is a rectangle.

§ 301, $\triangle$s $AJD$ and $EJG$ are similar;

$\therefore \quad AJ : AD = EJ : EG = 1 : 2$; $\therefore AJ = \frac{1}{2} AD$;

but, §§ 153, 198, $\quad AJ = DK = \frac{1}{2} DC$; $\therefore DC = AD$;

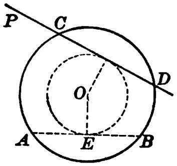
hence, § 153, $\quad AB = DC = AD = BC$.

Hence, $ABCD$ is a square.

Ex. **735.*** Through a given point draw a line so that it shall divide a given circumference into two parts having the ratio $3 : 7$.

**Solution.** § 405, divide the given circumference $ABD$ into 10 equal parts; take arc $AB$ equal to 3 of them and draw $AB$.

Draw $OE \perp AB$; with $O$ as center and radius $OE$, describe a ⊙; through the given point $P$ draw a tangent to this ⊙ intersecting the given circumference as in $C$ and $D$.



Then, $PCD$ is the required line.

**Proof.** § 202, chord $CD$ = chord $AB$;

hence, § 196, arc $CD$ = arc $AB$;

but, const., arc $AB = \frac{3}{10}$ of the circumference;

$\therefore$ arc $CD = \frac{3}{10}$ of the circumference.

Hence, arc $CD$ : arc $CABD = 3 : 7$.

Ex. **736.** Construct a circle equivalent to twice a given circle.

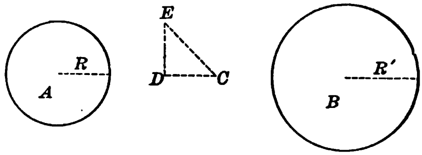
**Solution.** Denote the given ⊙ by $A$ and its radius by $R$.

Construct rt. $\angle CDE$ making $CD = DE = R$. Draw $CE$.

Construct ⊙ $B$ with radius

$R' = CE$.

Then, $B$ is the ⊙ required.



**Proof.** § 398, $A = \pi R^2$,

and $\quad 2A = 2\pi R^2 = \pi(R^2 + R^2) = \pi(\overline{CD}^2 + \overline{DE}^2) = \pi \overline{CE}^2 = \pi R'^2$;

but $\quad B = \pi R'^2$; hence, $B \Leftrightarrow 2A$.

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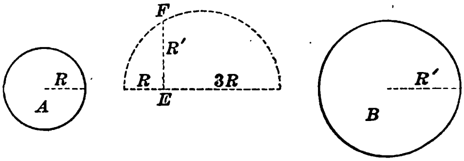
* See page 312.

**Ex. 737.** Construct a circle equivalent to three times a given circle.



**Solution.** Denote the given ⊙ by $A$ and its radius by $R$.

By Ex. 728, construct $EF = \sqrt{3}$, or $\sqrt{1 \times 3}$ when the unit is $R$.

Construct ⊙ $B$ with radius $R' = EF$.

Then, $B$ is the circle required.

**Proof.** $A = \pi R^2$. $B = \pi R'^2 = \pi(R\sqrt{3})^2 = 3\pi R^2$; hence, $B \Leftrightarrow 3A$.

## BOOK VII

Ex. **738.** What is the locus of the perpendiculars to a given straight line at a given point in the line?

**Solution.** Since, § 444, every perpendicular to a straight line at a given point lies in a plane which is perpendicular to the line at that point, the plane perpendicular to the given line at the given point is the required locus.

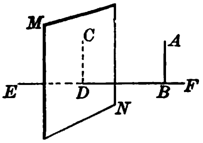
Ex. **739.** The length of a perpendicular from a given point to a plane is $5^{dm}$. What is the diameter of the circle which is the locus of the foot of an oblique line drawn from the same point to the plane, if the oblique line is $13^{dm}$ long?

**Solution.** The line drawn from the foot of the ⊥ to the foot of the oblique line is the radius of the circle and, § 443, forms a rt. △ with the ⊥ and the oblique line.

Then, § 350, this radius $= \sqrt{13^2 - 5^2} = \sqrt{144} = 12^{dm}$;

consequently, the diameter $= 12^{dm} \times 2 = 24^{dm}$.

Ex. **740.** If a straight line and a plane are perpendicular to another straight line, they are parallel.



**Data:** Plane $MN$ and line $AB$ each ⊥ $EF$.

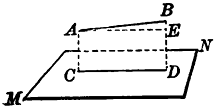
**To prove** $AB \parallel MN$.

**Proof.** Through $AB$ and $EF$ pass a plane intersecting $MN$ as in $CD$.

Then, § 443, $CD \perp EF$, but, data, $AB \perp EF$; ∴ § 71, $AB \parallel CD$.

Hence, § 457, $AB \parallel MN$.

Ex. **741.** If a line is equal to its projection on a plane, it is parallel to the plane.



**Data:** Line $AB$ equal to its projection $CD$ on the plane $MN$.

**To prove** $AB \parallel MN$.

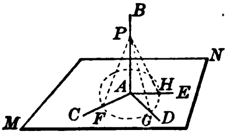
**Proof.** Draw $AE \parallel CD$.

Then, § 151, $AE = CD$; but, data, $AB = CD$; $\therefore AB = AE$.

But, since, § 72, $AE \perp BD$ and, § 61, is the shortest line from $A$ to $BD$, this is impossible unless $AB$ coincides with $AE$ and is therefore parallel to $CD$.

Then, $AB \parallel CD$; hence, § 457, $AB \parallel MN$.

**Ex. 742.** If a line makes equal angles with three intersecting lines in the same plane, it is perpendicular to that plane.



**Data**: Line $AB$ making equal $\angle$s with any three lines of plane $MN$, as $AC$, $AD$, and $AE$ at the point $A$.

**To prove** $AB \perp MN$.

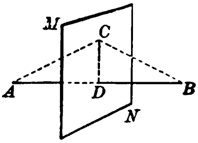
**Proof.** With $A$ as a center describe a circumference in $MN$, cutting $AC$, $AD$, and $AE$ in $F$, $G$, and $H$ respectively. Join $F$, $G$, and $H$ with any point $P$ of $AB$.

Then, § 100, $\triangle$s $FAP$, $GAP$, and $HAP$ are equal,

and $FP = GP = HP$; but $P$ is any point of $AB$.

Hence, § 450, $AB \perp MN$.

**Ex. 743.** If a plane bisects a straight line at right angles, any point in the plane is equidistant from the extremities of the line.



**Data**: Line $AB$; plane $MN \perp AB$ at its middle point $D$; and $C$, any point in $MN$.

**To prove** $C$ equidistant from $A$ and $B$.

**Proof.** Pass a plane through $C$ and $AB$ intersecting $MN$ as in $CD$.

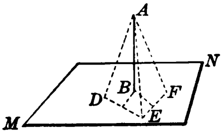
Then, § 443, $AB \perp CD$, and in $\triangle$s $ADC$ and $BDC$,

data, $AD = BD$, $CD$ is common, and, §§ 26, 52, $\angle ADC = \angle BDC$;

$\therefore$ § 100, $\triangle ADC = \triangle BDC$, and $AC = BC$;

that is, $C$ is equidistant from $A$ and $B$.

**Ex. 744.** Draw a perpendicular to a given plane from any point without the plane.



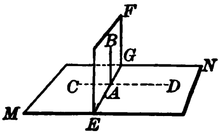
**Solution.** From the given point $A$ draw three equal oblique lines $AD$, $AE$, and $AF$ to the given plane $MN$.

In $MN$, Ex. 373, find the point $B$ equidistant from $D$, $E$, and $F$, and draw $AB$.

Then, $AB$ is the required perpendicular.

**Proof.** § 449, the foot of a $\perp$ from $A$ to $MN$ is equidistant from $D$, $E$, and $F$, and since $B$ is the only point in $MN$ which is equidistant from $D$, $E$, and $F$, it is the foot of the $\perp$ from $A$; that is, $AB \perp MN$.

**Ex. 745.** **Erect a perpendicular to a given plane at a given point in the plane.**



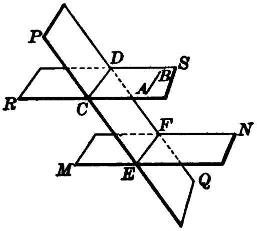
**Solution.** In the given plane $MN$ draw $CD$ through the given point $A$, and pass a plane $EF$ through $A$ perpendicular to $CD$, intersecting $MN$ as in $EAG$.

In the plane $EF$ draw $AB \perp EG$.

Then, $AB$ is the required perpendicular.

**Proof.** Const., $CD \perp EF$; $\therefore$ § 443, $CD \perp AB$; but, const., $AB \perp EG$; that is, $AB$ is perpendicular to two lines of $MN$; hence, § 442, $AB \perp MN$.

Ex. **746.** **A line parallel to two intersecting planes is parallel to their intersection.**



**Data:** Line $AB \parallel$ planes $MN$ and $PQ$ whose intersection is $EF$.

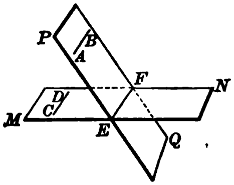
**To prove** $AB \parallel EF$.

**Proof.** Through $AB$ pass the plane $RS \parallel MN$ and intersecting $PQ$ as in $CD$.

Then, § 458, $AB \parallel CD$, and, § 463, $EF \parallel CD$;

hence, § 455, $AB \parallel EF$.

**Ex. 747.** **If two lines are parallel, the intersections of any planes passing through them are parallel.**



**Data:** Two parallel lines, as $AB$ and $CD$.

**To prove** that the intersections of any planes passed through $AB$ and $CD$ are parallel.

**Proof.** Pass any planes, as $MN$ and $PQ$, through $CD$ and $AB$ respectively, intersecting, as in $EF$.

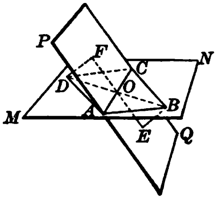
Then, § 457, $AB \parallel MN$, and $CD \parallel PQ$;

hence, § 458, $EF \parallel AB$ and $CD$.

Similarly, the intersections of any other planes passing through $AB$ and $CD$ are parallel to $AB$ and $CD$.

Therefore, § 455, these intersections are parallel to each other.

Ex. **748.** **If a plane is passed through a diagonal of a parallelogram, the perpendiculars to it from the extremities of the other diagonal are equal.**



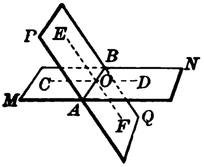
**Data:** ▱ $ABCD$ whose plane is $MN$ and the ⊥s $BE$ and $DF$ from the extremities of the diagonal $BD$ to $PQ$, any plane through diagonal $AC$.

**To prove** $BE = DF$.

**Proof.** Since, § 453, $BE \parallel DF$, these lines are in the same plane, and, § 427, $BD$ is also in the same plane; then, through $BE$, $DF$, and $BD$ pass a plane intersecting $PQ$ as in $FOE$.

Then, § 443, $BE$ and $DF \perp FOE$; $\therefore$ $\triangle$s $OEB$ and $OFD$ are rt. $\triangle$s,
§ 154, $OB = OD$, and, § 59, $\angle BOE = \angle DOF$;
hence, § 114, $\triangle OEB = \triangle OFD$, and $BE = DF$.

**Ex. 749.** If two planes intersect each other, the vertical dihedral angles are equal.



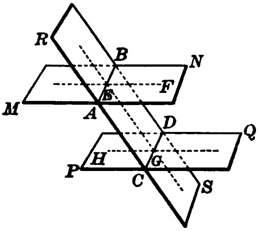
**Data:** Two planes, as $MN$ and $PQ$, intersecting in $AB$.

**To prove** vertical dihedral $\angle$s, as $M$-$AB$-$P$ and $Q$-$AB$-$N$, equal.

**Proof.** Through some point of $AB$, as $O$, draw $CD$ and $EF \perp AB$, and lying in planes $MN$ and $PQ$, respectively.

Then, § 473, $\angle$s $COE$ and $FOD$ are the plane $\angle$s of the dihedral $\angle$s $M$-$AB$-$P$ and $Q$-$AB$-$N$, respectively;
but, § 59, $\angle COE = \angle FOD$;
hence, § 477, dihedral $\angle M$-$AB$-$P$ = dihedral $\angle Q$-$AB$-$N$.

**Ex. 750.** If a plane intersects two parallel planes, the alternate interior dihedral angles are equal.



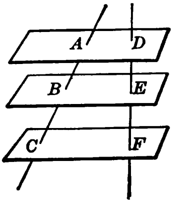
**Data:** Two parallel planes $MN$ and $PQ$ intersected by plane $RS$ in $AB$ and $CD$, respectively.

**To prove** the alternate interior dihedral $\angle$s, as $N$-$AB$-$S$ and $P$-$CD$-$R$, equal.

**Proof.** Pass a plane $\perp AB$ as at $E$, intersecting $MN$ in $EF$, $RS$ in $EG$, $PQ$ in $HG$, and $CD$ in $G$.

Then, since, § 463, $CD \parallel AB$, § 454, the plane $\perp AB$ is $\perp CD$; hence, § 443, $EF$ and $EG \perp AB$, and $EG$ and $HG \perp CD$; $\therefore$ § 473, $\angle$s $FEG$ and $EGH$ are the plane $\angle$s of the dihedrals $N$-$AB$-$S$ and $P$-$CD$-$R$, respectively; but since, § 463, $EF \parallel HG$, and $EG$ is a transversal,
§ 73, $\angle FEG = \angle EGH$;
hence, § 477, dihedral $\angle N$-$AB$-$S$ = dihedral $\angle P$-$CD$-$R$.

**Ex. 751.** The line $ABC$ pierces three parallel planes in $A$, $B$, and $C$, respectively, and the line $DEF$ pierces the same planes in $D$, $E$, and $F$, respectively. If $AB$ is 6 in., $BC$ 8 in., and $DF$ 12 in., what is the length of $DE$ and of $EF$?

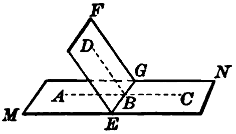


**Solution.** § 470, $AB : BC = DE : EF$;
$\therefore$ § 276, $AB + BC : BC = DE + EF : EF$;
that is, $14 : 8 = 12 : EF$;
whence, $EF = \frac{8 \times 12}{14} = 6\frac{6}{7}$ in.

Then, $DE = DF - EF = 12 \text{ in.} - 6\frac{6}{7} \text{ in.} = 5\frac{1}{7}$ in.

**Ex. 752.** If the sum of two adjacent dihedral angles is equal to two right dihedral angles, their exterior faces are in the same plane.

**Data:** Two adjacent dihedrals, as $M\text{-}EG\text{-}F$ and $F\text{-}EG\text{-}N$, whose sum is equal to two right dihedral ∠s.



**To prove** faces $MG$ and $EN$ in the same plane.

**Proof.** In planes $MG$, $EF$, and $EN$, respectively, draw $AB$, $DB$, and $CB \perp$ edge $EG$ as at $B$.

Then, § 479, plane ∠s $ABD$ and $DBC$ measure the dihedrals $M\text{-}EG\text{-}F$ and $F\text{-}EG\text{-}N$ respectively; consequently, $\angle ABD + \angle DBC = 2$ rt. ∠s; hence, § 58, $AB$ and $BC$ are in the same straight line.

Hence, $MG$ and $EN$ are in the same plane $MN$ determined by the two intersecting straight lines $AC$ and $EG$.

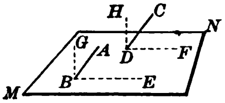
**Ex. 753.** Find the locus of all points equidistant from two parallel planes.

**Solution.** Since, § 440, a ⊥ measures the distance from a point to a plane, the middle point of any ⊥ to the given planes is equidistant from them, and since, § 462, a plane ⊥ to this line at its middle point is parallel to each of the given planes, § 465, every point in it is equidistant from the given planes, and every point not in it is at a greater distance from one of the given planes than from the other.

Hence, the locus of all points equidistant from two parallel planes is a plane parallel to them and midway between them.

**Ex. 754.** Parallel lines which pierce the same plane make equal angles with it.

**Data:** Parallel lines $AB$ and $CD$ piercing plane $MN$ in $B$ and $D$ respectively.



**To prove** that $AB$ and $CD$ make equal ∠s with $MN$.

**Proof.** Through $AB$ and $CD$, pass planes $\perp MN$, intersecting $MN$ in $BE$ and $DF$ respectively. In these planes draw $BG$ and $DH \perp BE$ and $DF$ respectively.

Then, § 480, $BG$ and $DH \perp MN$;

∴ § 453, $BG \parallel DH$,

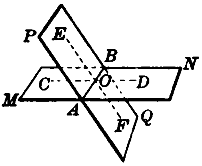
and since, data, $AB \parallel CD$,

§ 469, $\angle ABG = \angle CDH$;

hence, § 54, $\angle ABE = \angle CDF$;

that is, § 439, $AB$ and $CD$ make equal ∠s with $MN$.

**Ex. 755.** If two planes intersect each other, the sum of the two adjacent dihedral angles on the same side of either plane is equal to two right dihedral angles.



**Data:** Two planes, *MN* and *PQ*, intersecting in *AB*.

**To prove** the sum of two adjacent dihedral $\angle$s, as *M-AB-P* and *P-AB-N* = 2 rt. dihedral $\angle$s.

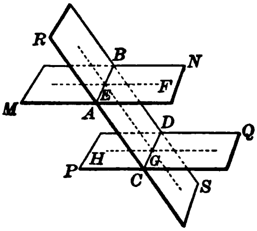
**Proof.** Through *O*, any point in *AB*, draw *CD* and *EF* $\perp$ *AB*, and lying in *MN* and *PQ* respectively.

Then, § 473, $\angle$s *COE* and *EOD* are the plane $\angle$s of the dihedrals *M-AB-P* and *P-AB-N* respectively, and, § 479, they are the measures of those dihedrals.

But, § 55, $\angle COE + \angle EOD = 2$ rt. $\angle$s;

hence, $M\text{-}AB\text{-}P + P\text{-}AB\text{-}N = 2$ rt. dihedral $\angle$s.

**Ex. 756.** If a plane intersects two parallel planes, the interior dihedral angles on the same side of the intersecting plane are supplementary.



**Data:** Plane *RS* intersecting two parallel planes *MN* and *PQ* in *AB* and *CD* respectively.

**To prove** two interior dihedrals on the same side of *RS*, as *M-AB-S* and *P-CD-R*, supplementary.

**Proof.** Ex. 755,

$\angle M\text{-}AB\text{-}S + \angle N\text{-}AB\text{-}S = 2$ rt. dihedral $\angle$s;

but, Ex. 750, $\angle N\text{-}AB\text{-}S = \angle P\text{-}CD\text{-}R$;

$\therefore$ substituting,

$\angle M\text{-}AB\text{-}S + \angle P\text{-}CD\text{-}R = 2$ rt. dihedral $\angle$s;

that is, § 475, dihedral $\angle$s *M-AB-S* and *P-CD-R* are supplementary.

**Ex. 757.** Find the locus of all points in space equidistant from two given points.

**Solution.** Since, § 452, the locus of a point in space equidistant from the extremities of a straight line is the plane $\perp$ to the line at its middle point, the locus of all points in space equidistant from two given points is the plane $\perp$ to the line which joins them at its middle point.

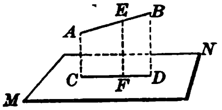
**Ex. 758.** Find the locus of all points at a given distance from a given plane.

**Solution.** Since, § 465, parallel planes are everywhere equally distant. the two planes (one on each side) parallel to the given plane and at the given distance from it constitute the required locus, for every point in each is at the given distance from the given plane, and every point not in either is at a distance greater or less than the given distance from the given plane.

**Ex. 759.** A line and its projection on a plane determine a second plane perpendicular to the first.

**Data:** Line $AB$ and its projection $CD$ on the plane $MN$.

**To prove** $AB$ and $CD$ determine a plane $\perp MN$.



**Proof.** Pass a plane through $AB$ and $CD$ and from any point in $AB$, as $E$, draw $EF \perp MN$.

Then, § 437, the foot of $EF$ is in $CD$,

and, § 427, $EF$ lies in plane $CB$.

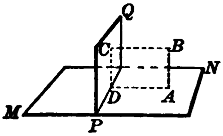
Hence, § 482, plane $CB \perp MN$;

that is, $AB$ and $CD$ determine a plane $\perp MN$.

**Ex. 760.** If a line is parallel to one plane and perpendicular to another, the two planes are perpendicular to each other.

**Data:** Line $AB$, which is parallel to plane $PQ$ and perpendicular to plane $MN$.

**To prove** $PQ \perp MN$.

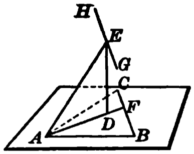


**Proof.** Pass a plane through $AB$ intersecting $PQ$ as in $CD$.

Then, § 458, $CD \parallel AB$; $\therefore$ § 454, $CD \perp MN$.

Hence, § 482, $PQ \perp MN$.

**Ex. 761.** $D$ is any point in the perpendicular $AF$ from $A$ to the side $BC$ of the triangle $ABC$. If $DE$ is perpendicular to the plane $ABC$, and $GH$ passing through $E$ is parallel to $BC$, then, $AE$ is perpendicular to $GH$.



**Proof.** Pass a plane through $ADE$.

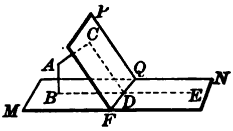
Then, § 482, plane $ADE \perp$ plane $ABC$ and intersects it in $AF$.

Since $BC \perp AF$, § 480, $BC \perp$ plane $ADE$;

but, data, $GH \parallel BC$;

$\therefore$ § 454, $GH \perp$ plane $ADE$; hence, § 443, $GH \perp AE$; that is, $AE \perp GH$.

**Ex. 762.** If from a point within a dihedral angle perpendiculars are drawn to its faces, the angle contained by these perpendiculars is equal to the plane angle of the adjacent dihedral angle formed by producing one of the planes.



**Data:** Dihedral $\angle M\text{-}FQ\text{-}P$; a point $A$ within it; and the $\perp$s $AB$ and $AC$ from $A$ to the faces; also the adjacent dihedral $\angle P\text{-}FQ\text{-}N$, formed by producing plane $MQ$.

**To prove** $\angle BAC =$ plane $\angle$ of dihedral $\angle P\text{-}FQ\text{-}N$.

**Proof.** Through $AB$ and $AC$ pass a plane intersecting $PQ$ in $CD$, $MN$ in $BDE$, and $FQ$ in $D$.

Then, § 482, plane $DBAC \perp MN$ and $PQ$; $\therefore$ § 485, plane $DBAC \perp FQ$; hence, § 443, $BD$, $CD$, and $ED \perp FQ$;

$\therefore$ § 473, $\angle$s $BDC$ and $CDE$ are the plane $\angle$s of the dihedral $\angle$s $M\text{–}FQ\text{–}P$ and $P\text{–}FQ\text{–}N$ respectively.

Now, since in the quadrilateral $DBAC$, $\angle$s $B$ and $C$ are rt. $\angle$s,

§ 32, $\angle BAC$ is the supplement of $\angle BDC$;

but, § 55, $\angle CDE$ is the supplement of $\angle BDC$;

$\therefore$ § 54, $\angle BAC = \angle CDE$, the plane $\angle$ of dihedral $\angle P\text{–}FQ\text{–}N$.

Ex. **763**. The shortest line that can be drawn between two straight lines not in the same plane is their common perpendicular.

**Data**: (See Fig. for § 490.) Two straight lines $AB$ and $CD$ not in the same plane; $DH$ their common $\perp$; and any other line between them as $JK$.

**To prove** $DH < JK$.

**Proof.** Assuming the same construction as in § 490, it has been shown that $JE \perp MN$, and $JK$ is not perpendicular to $MN$.

Then, § 448, $JE < JK$; but, § 151, $DH = JE$; hence, $DH < JK$.

Ex. **764**. A plane can be passed perpendicular to only one edge and only two faces of a polyhedral angle.

**Proof.** If a plane could be perpendicular to two edges of a polyhedral angle there would be two $\perp$s from a point to a plane, which, § 456, is impossible.

Hence, a plane can be passed perpendicular to only one edge of a polyhedral angle.

Since a plane $\perp$ to two faces is $\perp$ to their intersection, a plane $\perp$ to more than two faces would be $\perp$ to more than one intersection, or edge; but it has been shown that this is impossible.

Therefore, a plane can be passed perpendicular to only one edge and only two faces of a polyhedral angle.

Ex. **765**. Every point within a dihedral and equidistant from its faces lies in the plane which bisects that dihedral.

**Data**: (See Fig. for § 488.) Dihedral $\angle C\text{–}AB\text{–}D$, and any point, as $H$, within it and equidistant from its faces.

**To prove** that $H$ lies in the plane that bisects dihedral $C\text{–}AB\text{–}D$.

**Proof.** Through $H$ and $AB$ pass the plane $AK$; draw $HE$ and $HF \perp$ faces $CB$ and $AD$ respectively; and pass a plane through $HE$ and $HF$ intersecting $CB$, $AK$, and $AD$ as in $EG$, $HG$, and $FG$ respectively.

Then, § 482, plane $EFGH \perp CB$ and $AD$;

$\therefore$ § 485, $EFGH \perp AB$;

hence, § 443, $EG$, $HG$, and $FG \perp AB$,

and, § 473, $\angle$s $EGH$ and $FGH$ are the plane $\angle$s of the dihedral $\angle$s $C\text{–}AB\text{–}K$ and $D\text{–}AB\text{–}K$.

But in rt. $\triangle$s $HEG$ and $HFG$, $HE = HF$, and $HG$ is common;

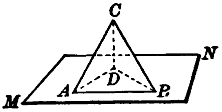
$\therefore$ § 123, $\triangle HEG = \triangle HFG$, and $\angle EGH = \angle FGH$;

hence, § 477, dihedral $\angle C\text{-}AB\text{-}K =$ dihedral $\angle D\text{-}AB\text{-}K$;

that is, plane $AK$ bisects dihedral $\angle C\text{-}AB\text{-}D$.

Therefore, $H$ lies in the plane that bisects dihedral $\angle C\text{-}AB\text{-}D$.

Ex. **766.** The sides of an isosceles triangle are equally inclined to any plane through its base.



**Data**: Isos. $\triangle ABC$ and any plane $MN$ through its base $AB$.

**To prove** $AC$ and $BC$ equally inclined to $MN$.

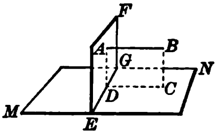
**Proof.** Draw $CD \perp MN$ and pass the planes $ACD$ and $BCD$.

§ 482, these planes are perpendicular to $MN$, and their intersections $AD$ and $BD$ with $MN$ are the projections respectively of $AC$ and $BC$ upon $MN$.

§ 443, $CD \perp AD$ and $BD$; $\therefore$ in rt. $\triangle$s $ADC$ and $BDC$,

data, $AC = BC$, and $CD$ is common; hence, § 123, $\triangle ADC = \triangle BDC$, and $\angle CAD = \angle CBD$; that is, § 439, $AC$ and $BC$ are equally inclined to $MN$.

Ex. **767.** If a straight line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.



**Data**: Line $AB \parallel$ plane $MN$, and plane $EF \perp AB$.

**To prove** $EF \perp MN$.

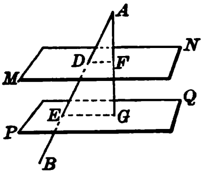
**Proof.** From $A$, the point of intersection of $AB$ and $EF$, draw $AD \perp EG$, the intersection of $EF$ and $MN$, and through $AD$ and $AB$ pass a plane intersecting $MN$ as in $DC$.

Then, § 458, $DC \parallel AB$; but, § 443, $AD \perp AB$;

hence, § 72, $AD \perp DC$, and since, const., $AD \perp EG$,

§ 442, $AD \perp MN$; hence, § 482, $EF \perp MN$.

Ex. **768.** If a straight line intersects two parallel planes it makes equal angles with them.



**Data**: The straight line $AB$ intersecting the parallel planes $MN$ and $PQ$ in $D$ and $E$ respectively.

**To prove** $AB$ makes equal $\angle$s with $MN$ and $PQ$.

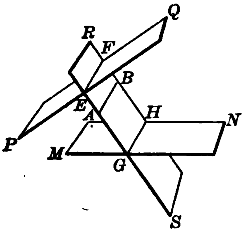
**Proof.** From any point of $AB$ as $A$ draw $AG \perp PQ$ at $G$, and piercing $MN$ in $F$; through $AB$ and $AG$ pass a plane intersecting $MN$ in $DF$ and $PQ$ in $EG$.

Since $AG \perp PQ$, § 466, $AG \perp MN$; hence, § 482, plane $ABG \perp MN$ and $PQ$, and $DF$ and $EG$ are in the projections of $AB$ upon $MN$ and $PQ$ respectively.

§ 463, $DF \parallel EG$; hence, § 76, $\angle ADF = \angle AEG$;

that is, § 439, $AB$ makes equal $\angle$s with $MN$ and $PQ$.

**Ex. 769.** If a line is parallel to each of two planes, the intersections which any plane passing through it makes with the planes are parallel.



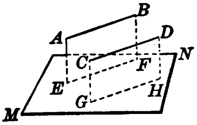
**Data:** Line $AB \parallel$ planes $MN$ and $PQ$, and any plane, as $RS$, passing through $AB$ and intersecting $MN$ and $PQ$ in $GH$ and $EF$ respectively.

**To prove** $GH \parallel EF$.

**Proof.** § 458, $GH \parallel AB$, and $EF \parallel AB$;

hence, § 80, $GH \parallel EF$.

Ex. **770.** The projections of parallel straight lines on any plane are either parallel or coincident.



**Data:** Two parallel straight lines $AB$ and $CD$ and their projections $EF$ and $GH$ respectively on plane $MN$.

**To prove** $EF$ either parallel to or coincident with $GH$.

**Proof.** (1) When $AB$ and $CD$ do not lie in the same plane, $\perp MN$. If $EF$ and $GH$ are not parallel, they would meet, if produced, in some point which we may designate $O$, and from $O$ there could be drawn in planes $AF$ and $CH$ respectively lines parallel to $AB$ and $CD$ respectively; but, § 80, the line parallel to $AB$ would also be parallel to $CD$, and, therefore, these two lines would both be parallel to $CD$, and, consequently, parallel to each other from the same point $O$, which is impossible.

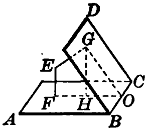
Hence, the supposition that $EF$ and $GH$ are not parallel is untenable; consequently, $EF \parallel GH$.

(2) When $AB$ and $CD$ lie in the same plane perpendicular to $MN$, then their projections $EF$ and $GH$ coincide.

Ex. **771.** Find the locus of points which are equidistant from three given points not in the same straight line.

**Solution.** Since, § 207, a circumference may be passod through the given points, § 450, the $\perp$ to the plane of the points at the center of the circumference passing through them is the required locus.

Ex. **772.** From any point within the dihedral angle $A$-$BC$-$D$, $EF$ and $EG$ are drawn perpendicular to the faces $AC$ and $BD$, respectively, and $GH$ perpendicular to $AC$ at $H$. Prove that $FH$ is perpendicular to $BC$.



**Proof.** Pass a plane through $EF$ and $EG$ intersecting $AC$ as in $FO$ and $CD$ as in $GO$.

Then, § 482, plane $OFEG \perp AC$ and $BD$;

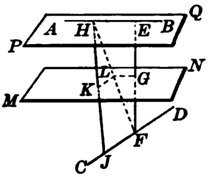
$\therefore$ § 485, plane $OFEG \perp BC$; hence, § 443, $FO \perp BC$.

In plane $OFEG$ draw a line from $G \perp FO$.

Then, § 480, this line is perpendicular to $AC$, and it must coincide with $GH$, for, § 456, from a point only one perpendicular can be drawn to a plane; hence, $H$ is in $FO$; but $FO \perp BC$;

therefore, $FH \perp BC$.

Ex. **773.** If a plane is passed through the middle point of the common perpendicular to two straight lines in space, and parallel to both lines, it bisects every straight line drawn from any point in one line to any point in the other line.



**Data:** Straight lines $AB$ and $CD$; their common $\perp$, $EF$; plane $MN \parallel AB$ and $CD$ through $G$, the middle point of $EF$; and any line, as $HJ$, drawn from $AB$ to $CD$ and cut by $MN$ in the point $K$.

**To prove** $HJ$ bisected at $K$.

**Proof.** Draw $HF$ piercing $MN$ as in $L$.

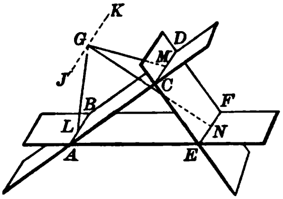
Then, plane $HFE$ intersects $MN$ in $LG$, and plane $JFH$ intersects $MN$ in $KL$.

Pass the plane $PQ \parallel MN$ through $AB$.

Then, in $\triangle HFE$, § 463, $LG \parallel HE$, and since $G$ is the middle point of $EF$, § 158, $L$ is the middle point of $HF$.

Similarly, by passing a plane $\parallel MN$ through $CD$, it may be shown that $KL \parallel JF$, and since $L$ is the middle point of $HF$, $HJ$ is bisected at $K$.

Ex. **774.** If the intersections of several planes are parallel, the perpendiculars drawn to them from any point lie in one plane.



**Data:** $GL$, $GM$, $GN$, etc., $\perp$s from $G$ to the intersections, $AB$, $CD$, $EF$, etc., of several planes, these intersections being parallel to each other.

**To prove** $GL$, $GM$, $GN$, etc., lie in one plane.

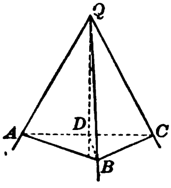
**Proof.** Through $G$ draw $JK \parallel AB$.

Then, § 80, $JK \parallel CD$, $EF$, etc.;

$\therefore$ § 72, $GL$, $GM$, $GN$, etc., $\perp JK$;

hence, § 444, $GL$, $GM$, $GN$, etc., lie in one plane.

Ex. **775.** If two face angles of a trihedral are equal, the dihedral angles opposite them are also equal.



**Data:**

Trihedral $Q\text{-}ABC$ in which face $\angle AQB =$ face $\angle BQC$.

**To prove** dihedral $\angle AQ =$ dihedral $\angle CQ$.

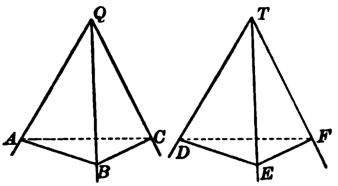
**Proof.** In face $AQC$ draw $QD$ to bisect face $\angle AQC$, and through $B$, $Q$, and $D$ pass a plane dividing $Q\text{-}ABC$ into the trihedrals $Q\text{-}ABD$ and $Q\text{-}BCD$.

In these trihedrals, data, face $\angle AQB =$ face $\angle BQC$,
const., face $\angle AQD =$ face $\angle DQC$, and face $\angle BQD$ is common;
hence, § 501, dihedral $\angle AQ =$ dihedral $\angle CQ$.

Ex. **776.** **A trihedral angle, having two of its dihedral angles equal, may be made to coincide with its symmetrical trihedral angle.**

**Data:** Trihedral $\angle$ *Q–ABC* having dihedral $\angle$s *AQ* and *CQ* equal; and trihedral $\angle$ *T-DEF* symmetrical to *Q–ABC*.

**To prove** that *Q–ABC* may be made to coincide with *T-DEF*.



**Proof.** Apply *Q–ABC* to *T-DEF* so that the equal face $\triangle AQC$ and $\triangle DTF$ shall coincide.

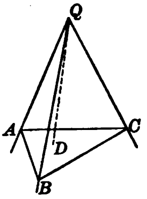
Since, data, § 494, dihedrals *AQ*, *CQ*, *DT*, and *FT* are equal, the planes of faces *AQB* and *CQB* coincide respectively with the planes of faces *DTE* and *FTE*; hence, the intersections, *BQ* and *ET*, of these pairs of planes must coincide, and the dihedrals *BQ* and *ET* coincide.

Consequently, *Q–ABC* may be made to coincide with *T-DEF*.

Ex. **777.** **In any trihedral the three planes bisecting the three dihedrals intersect in the same straight line.**

**Data:** Trihedral $\angle$ *Q–ABC* whose dihedrals are *AQ*, *BQ*, and *CQ*; and the planes bisecting these dihedrals.

**To prove** that the planes bisecting dihedrals *AQ*, *BQ*, and *CQ* intersect in the same straight line.



**Proof.** The planes bisecting two of the dihedrals, as *AQ* and *BQ*, intersect in some line, as *DQ*.

§ 488, every point of *DQ* is equidistant from faces *AQB* and *AQC*, and also from faces *AQB* and *BQC*; hence, every point in *DQ* is equidistant from faces *AQC* and *BQC*.

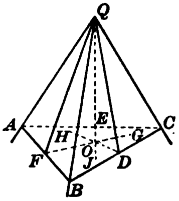
Hence, Ex. 765, *DQ* is in the plane that bisects dihedral *CQ*.

Consequently, the planes bisecting dihedrals *AQ*, *BQ*, and *CQ* intersect in the same straight line *DQ*.

Ex. **778.** **In any trihedral the planes which bisect the three face angles, and are perpendicular to the faces respectively, intersect in the same straight line.**

**Data:** Any trihedral, as *Q–ABC*, and the planes perpendicular to the faces, and intersecting them in the lines *QF*, *QE*, *QD*, which respectively bisect the face angles.

**To prove** that these bisecting planes intersect in the same straight line.



**Proof.** On the edges of the trihedral measure off the equal distances $QA$, $QB$, and $QC$, and through $A$, $B$, and $C$ pass a plane.

Then, § 90, ≜ $ABQ$, $BCQ$, and $ACQ$ are isosceles; ∴ § 120, $QF$, $QD$, and $QE$ are the perpendicular bisectors of $AB$, $BC$, and $AC$, respectively; hence, § 480, the bisecting planes are also perpendicular to $AB$, $BC$, and $AC$, and, § 443, their intersections $FG$, $DH$, and $EJ$, with plane $ABC$, are the perpendicular bisectors of the sides of $\triangle ABC$. But, § 170, $FG$, $DH$, and $EJ$ meet in a point, as $O$; ∴ $Q$ and $O$ are common to the three bisecting planes; hence, §§ 427, 436, the bisecting planes intersect in the same straight line $QO$.

## BOOK VIII

Ex. **779.** The perimeter of the base of a regular pyramid is 14 ft., and its slant height is 6 ft. What is its lateral area?

**Solution.** § 551, lat. area $= \frac{1}{2} \times 14 \times 6 = 42$ sq. ft.

Ex. **780.** What is the lateral area of a right prism whose altitude is 12 in., and the perimeter of whose base is 20 in.?

**Solution.** § 520, lat. area $= 12 \times 20 = 240$ sq. in.

Ex. **781.** Find the ratio of two rectangular parallelopipeds, if their altitudes are each $7^{m}$, and their bases $3^{m}$ by $4^{m}$ and $7^{m}$ by $9^{m}$, respectively.

**Solution.** § 534, the ratio of the parallelopipeds is $3 \times 4 : 7 \times 9 = 4 : 21$.

Ex. **782.** Find the ratio of two rectangular parallelopipeds, if their dimensions are $2^{dm}$, $4^{dm}$, $3^{dm}$, and $6^{dm}$, $7^{dm}$, $8^{dm}$, respectively.

**Solution.** § 535,

the ratio of the parallelopipeds is $2 \times 4 \times 3 : 6 \times 7 \times 8 = 1 : 14$.

Ex. **783.** What is the volume of a rectangular parallelopiped whose edges are $20.5^{m}$, $12.75^{m}$, and $8.6^{m}$, respectively?

**Solution.** § 536, vol. $= 20.5 \times 12.75 \times 8.6 = 2247.825^{sq\ m}$.

Ex. **784.** The altitude of a regular hexagonal prism is 12 ft., and each side of its base is 10 ft. What is its volume?

**Solution.** Ex. 648, the apothem of the base $= \frac{1}{2} \cdot 10\sqrt{3} = 5\sqrt{3}$.

Perimeter of the base $= 6 \times 10 = 60$;

∴ § 389, area of base $= \frac{1}{2} \times 60 \times 5\sqrt{3} = 150\sqrt{3}$;

hence, § 543, vol. of prism $= 150\sqrt{3} \times 12 = 1800\sqrt{3} = 3117.6+$ cu. ft.

Ex. **785.** What is the volume of a pyramid whose altitude is $18^{dm}$, and whose base is a rectangle $10^{dm}$ by $6^{dm}$?

**Solution.** § 330, area of base $= 10 \times 6 = 60^{sq\ dm}$.

Then, § 560, vol. of pyramid $= \frac{1}{3} \times 60 \times 18 = 360^{cu\ dm}$.

Ex. **786**. What is the volume of a truncated right triangular prism, if each side of its base is 3 ft., and its edges are 3 ft., 4 ft., and 6 ft., respectively?

**Solution.** Ex. 632,

$$\text{area of base} = \sqrt{4\tfrac{1}{2} \times 1\tfrac{1}{2} \times 1\tfrac{1}{2} \times 1\tfrac{1}{2}} = \tfrac{9}{4}\sqrt{3} \text{ sq. ft.}$$

Then, § 566,

$$\text{vol. of prism} = \tfrac{9}{4}\sqrt{3} \times \tfrac{1}{3}(3 + 4 + 6) = \tfrac{39}{4}\sqrt{3} = 16.887+ \text{ cu. ft.}$$

Ex. **787**. What is the lateral area of the frustum of a square pyramid whose slant height is $13^{\text{m}}$, each side of the lower base being $3.5^{\text{m}}$, and of the upper base $2^{\text{m}}$?

**Solution.** Perimeter of lower base $= 3.5^{\text{m}} \times 4 = 14^{\text{m}}$,

and perimeter of upper base $= 2^{\text{m}} \times 4 = 8^{\text{m}}$;

hence, § 552, lat. area of frustum $= \frac{1}{2} \times 13 \times (14 + 8) = 143^{\text{sq m}}$.

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**Proof.** 1. By const., § 450, the four faces of *D–ABC* are equal equilateral $\triangle$; hence, § 117, all the face $\angle$ are equal; consequently, § 500, the trihedral $\angle$ *A*, *B*, *C*, and *D* are equal; hence, § 577, *D–ABC* is a regular tetrahedron.

2. Const., the faces are equal squares; hence, all the face $\angle$ are equal; consequently, § 500, the trihedral $\angle$ *A*, *B*, *C*, *D*, *E*, *F*, *G*, and *H* are equal, and, § 577, *AG* is a regular hexahedron.

3. § 450, all the lines from *E* and *F* to *A*, *B*, *C*, and *D* are equal and each is equal to *AB*; hence, the eight faces of *E–ABCD–F* are equal equilateral $\triangle$.

Const., *O* is the center of the square *ABCD*; hence, the diagonal passes through *O*, and the lines *EF* and *BD* intersect at *O* and are in the same plane; consequently, *F*, *B*, *E*, and *D* are in one plane.

In the $\triangle$ *BED*, *BAD*, and *BFD*, the side *BD* is common, and the other sides are all equal; hence, § 107, the $\triangle$ are all equal and since $\angle$ *BAD* is a rt. $\angle$, $\angle$ *BED* and *BFD* are rt. $\angle$; similarly, it may be shown that the other $\angle$ of *FBED* are rt. $\angle$; consequently, *FBED* is a square equal to *ABCD*.

Then, the pyramids *A–FBED* and *E–ABCD* have their bases and faces respectively equal; hence, they can be made to coincide and are equal; consequently, polyhedral $\angle A =$ polyhedral $\angle E$.

Similarly, any two polyhedral $\angle$ are equal.

Hence, § 577, *E–ABCD–F* is a regular octahedron.

4. The surface constructed is composed of twelve equal regular pentagons; then, § 374, the face $\angle$ of each of its trihedral $\angle$ are equal to the face $\angle$ of every other one of its trihedral $\angle$; consequently, § 500, the trihedral $\angle$ are all equal, and, § 577, the figure is a regular dodecahedron.

5. Const., the twenty faces are equal equilateral $\triangle$s.

Since the five face $\angle$s of any one of the polyhedrals are equal to each other and to the face $\angle$s of any other of the polyhedrals, the polyhedrals may be made to coincide and are equal.

Hence, the figure is a regular icosahedron.

Ex. **788**. The edge of a cube is $5\frac{1}{2}$ in. What are its volume and surface?

**Solution.** § 536, $V = d \times e \times f = 5\frac{1}{2} \times 5\frac{1}{2} \times 5\frac{1}{2} = 166\frac{3}{8}$ cu. in.

§ 330, area of surface $= 6 \times (5\frac{1}{2} \times 5\frac{1}{2}) = 181\frac{1}{2}$ sq. in.

Ex. **789**. What are the entire area and volume of a right prism 4.5 ft. in altitude, if the bases are equilateral triangles 13 in. on a side?

**Solution.** $P = 13$ in. $\times 3 = 39$ in., or $3\frac{1}{4}$ ft.;

then, §§ 583, 520, $A = H \times P = 4.5 \times 3\frac{1}{4} = 14.625$ sq. ft.,

and, Ex. 632, area of base $= \sqrt{\frac{39}{24} \times \frac{13}{24} \times \frac{13}{24} \times \frac{13}{24}} = \frac{13^2}{24^2}\sqrt{3} = .508+$ sq. ft.;

hence, total area $= .508+$ sq. ft. $+ .508+$ sq. ft. $+ 14.625+$ sq. ft. $= 15.641+$ sq. ft.

§§ 583, 542, $V = B \times H = .508+$ sq. ft. $\times 4.5 = 2.286+$ cu. ft.

Ex. **790**. What is the total area of a regular triangular pyramid whose slant height is $15^{\text{dm}}$ and each side of whose base is $9^{\text{dm}}$?

**Solution.** Ex. 632, area of base $= \sqrt{\frac{27}{2} \times \frac{9}{2} \times \frac{9}{2} \times \frac{9}{2}} = \frac{81}{4}\sqrt{3} = 35.073+^{\text{sq dm}}$.

§§ 583, 551, $A = \frac{1}{2} P \times L = \frac{1}{2} \times 27 \times 15 = 202.5^{\text{sq dm}}$.

Then, total area $= 202.5^{\text{sq dm}} + 35.073+^{\text{sq dm}} = 237.573+^{\text{sq dm}}$.

Ex. **791**. What is the volume of a triangular pyramid whose altitude is 11 ft. and the sides of whose base are 3 ft., 4 ft., and 5 ft.?

**Solution.** Ex. 632, area of base $= \sqrt{6 \times 3 \times 2 \times 1} = 6$ sq. ft.;

hence, §§ 583, 559, $V = \frac{1}{3} B \times H = \frac{1}{3} \times 6 \times 11 = 22$ cu. ft.

Ex. **792**. What is the edge of a cube whose volume equals that of a rectangular parallelopiped whose edges are 9 in., 12 in., and 16 in.?

**Solution.** § 536, $V = d \times e \times f = 9 \times 12 \times 16 = 1728$ cu. in.

Then, edge of cube of equal volume $= \sqrt[3]{1728} = 12$ in.

Ex. **793**. The altitude of a prism is $6^{\text{dm}}$ and the area of its base is $2.5^{\text{sq dm}}$. What is the altitude of a prism of the same volume, if the area of its base is $3.75^{\text{sq dm}}$?

**Solution.** §§ 583, 543, $V = B \times H = 2.5 \times 6 = 15^{\text{cu dm}}$.

Then, in second prism, $V = B \times H = 3.75 \times H = 15$;

whence, $$H = \frac{15}{3.75} = 4^{\text{dm}}.$$

Ex. **794**. The homologous edges of two similar tetrahedrons are as 4 : 5. What is the ratio of their surfaces? What is the ratio of their volumes?

**Solution.** § 572, the ratio of their surfaces $= 4^2 : 5^2 = 16 : 25$.

§ 575, the ratio of their volumes $= 4^3 : 5^3 = 64 : 125$.

Ex. **795**. What is the altitude of a pyramid whose volume is $36^{\text{cu m}}$ and the sides of whose triangular base are $6^{\text{m}}$, $5^{\text{m}}$, and $4^{\text{m}}$?

**Solution.** Ex. 632, area of base $= \sqrt{\frac{15}{2} \times \frac{3}{2} \times \frac{5}{2} \times \frac{7}{2}} = \frac{15}{4}\sqrt{7}^{\text{sq m}}$.

Then, §§ 583, 559, $V = \frac{1}{3}B \times H = \frac{1}{3} \times \frac{15}{4}\sqrt{7} \times H = 36$;

whence, $$H = \frac{4 \times 36}{5\sqrt{7}} = 10.885+^{\text{m}}.$$

Ex. **796**. The area of the upper base of the frustum of a pyramid is 48 sq. ft., and that of the lower base is 72 sq. ft. If the altitude of the frustum is 60 ft., what is its volume?

**Solution.** §§ 583, 564,
$$\begin{aligned} V &= \tfrac{1}{3}H(B + b + \sqrt{B \times b}) \\ &= \tfrac{1}{3} \times 60(72 + 48 + \sqrt{72 \times 48}) \\ &= 20 \times 178.78775+ = 3575.755+ \text{ cu. ft.} \end{aligned}$$

Ex. **797**. What is the altitude of the frustum of a regular hexagonal pyramid, if its volume is $16^{\text{cu m}}$ and the sides of its bases are respectively $1.5^{\text{m}}$ and $2.5^{\text{m}}$?

**Solution.** Ex. 648, apothem of lower base $= \frac{2.5}{2}\sqrt{3}$;

hence, § 389, area of lower base $= \frac{1}{2} \times 15 \times \frac{2.5}{2}\sqrt{3} = 9.375\sqrt{3}$.

Similarly, area of upper base $= \frac{1}{2} \times 9 \times \frac{1.5}{2}\sqrt{3} = 3.375\sqrt{3}$.

Then, §§ 583, 564,
$$V = \tfrac{1}{3}H\left(9.375\sqrt{3} + 3.375\sqrt{3} + \sqrt{9.375\sqrt{3} \times 3.375\sqrt{3}}\right) = 16^{\text{cu m}};$$

that is, $$\frac{H}{\sqrt{3}}(12.75 + 5.625) = 16. \quad \therefore H = \frac{16\sqrt{3}}{18.375} = 1.508+^{\text{m}}.$$

Ex. **798**. A pyramid 20 ft. high has 100 sq. ft. in its base; a section parallel to the base has an area of 55 sq. ft. How far is the section from the base?

**Solution.** Let $x$ denote the distance of the section from the vertex.

Then, § 554, $55 : 100 = x^2 : 20^2$;

whence, $$x^2 = \frac{55 \times 20^2}{100} = 220, \text{ and } x = \sqrt{220} = 14.83+ \text{ ft.}$$

consequently, the section is 20 ft. − 14.83+ ft. = 5.17− ft. from the base.

Ex. **799**. What is the volume of an oblique truncated triangular prism whose edges are $5^{\text{m}}$, $7^{\text{m}}$, and $9^{\text{m}}$, and the area of whose right section is $16^{\text{sq m}}$?

**Solution.** § 567, $V = 16 \times \frac{1}{3}(5 + 7 + 9) = 112^{\text{cu m}}$.

**Ex. 800.** What is the edge of a cube whose entire area is $1^{\text{sq m}}$?

**Solution.** Area of one face $= \frac{1}{6}$ of a square meter;

then, the edge $= \sqrt{\frac{1}{6}} = .408247+^{\text{m}}$.

**Ex. 801.** The base of a pyramid contains 121 sq. ft.; a section parallel to the base, and 3 ft. from the vertex, contains 49 sq. ft. What is the altitude of the pyramid?

**Solution.** Denote the altitude by $h$.

Then, § 554, $49 : 121 = 3^2 : h^2$;

whence, $h^2 = \frac{121 \times 3^2}{49}$, and $h = \sqrt{\frac{121 \times 3^2}{49}} = \frac{33}{7} = 4\frac{5}{7}$ ft.

**Ex. 802.** What is the lateral area of a regular hexagonal pyramid whose base is inscribed in a circle whose diameter is 15 ft., the altitude of the pyramid being 8 ft.? What is the volume of the pyramid?

**Solution.** Ex. 648, side of base $= r = \frac{1}{2}$ of $15 = 7\frac{1}{2}$ ft.,

and apothem of base $= \frac{1}{2} r\sqrt{3} = \frac{15}{4}\sqrt{3}$;

hence, slant height $= \sqrt{(\frac{15}{4}\sqrt{3})^2 + 8^2} = 10.3047+$ ft.

Then, §§ 583, 551, $A = \frac{1}{2} P \times L = \frac{1}{2} \times 45 \times 10.3047+ = 231.85575+$ sq. ft.

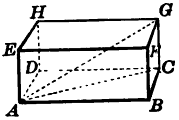
§ 389, area of base $= \frac{1}{2} \times 45 \times \frac{15}{4}\sqrt{3} = 146.1375+$ sq. ft.;

hence, §§ 583, 560, $V = \frac{1}{3} B \times H = \frac{1}{3} \times 146.1375+ \times 8 = 389.7+$ cu. ft.

**Ex. 803.** Any lateral edge of a right prism is equal to the altitude.

**Proof.** § 507, the lateral edges are perpendicular to the base, also a line representing the altitude is perpendicular to the base; hence, § 453, any lateral edge is parallel to a line representing the altitude; then, since, § 505, the bases are parallel planes, § 464, any lateral edge is equal to the altitude.

**Ex. 804.** The square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares of its three edges.



**Data:** Rectangular parallelopiped $BH$, whose dimensions are $AB$, $BC$, and $CG$; and any diagonal, as $AG$.

**To prove** $\overline{AG}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CG}^2$.

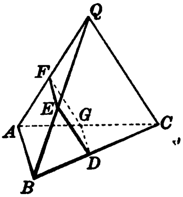
**Proof.** § 349, $\overline{AG}^2 = \overline{AC}^2 + \overline{CG}^2$;

but $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$; $\therefore$ substituting, $\overline{AG}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CG}^2$.

**Ex. 805.** If the edges of a tetrahedron are all equal, the sum of the angles at any corner is equal to two right angles.

**Proof.** Since the edges are all equal, each face is an equilateral △, and, Ex. 74, each face ∠ contains 60°; since there are three angles at any corner, their sum is 60° × 3 = 180°, or 2 rt. ∠s.

Ex. **806**. The section of a triangular pyramid made by a plane parallel to two opposite edges is a parallelogram.



**Data**: Triangular pyramid $Q\text{-}ABC$, and the section $DEFG \parallel AB$ and $QC$.

**To prove** $DEFG$ a parallelogram.

**Proof.** § 458, $ED$ and $FG \parallel QC$; $\therefore$ § 455, $ED \parallel FG$.

Similarly, $EF \parallel DG$, each being parallel to $AB$.

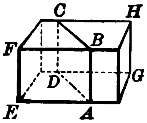
Hence, § 140, $DEFG$ is a parallelogram.

Ex. **807**. The lateral faces of right prisms are rectangles.

**Proof.** Since, § 507, the lateral edges of a right prism are perpendicular to its bases, § 443, they are perpendicular to the sides of the bases; that is, the $\angle$s of the lateral faces are rt. $\angle$s; hence, §§ 505, 141, the lateral faces are rectangles.

Ex. **808**. The section of a prism made by a plane parallel to a lateral edge is a parallelogram.

**Data**: Prism $EH$ and a section of it, as $ABCD \parallel$ edge $EF$.



**To prove** $ABCD$ a parallelogram.

**Proof.** Since, data, $EF \parallel$ plane $ABCD$, § 458, $AB \parallel EF$;

similarly, $DC \parallel GH$; but, § 505, $EF \parallel GH$;

$\therefore$ § 455, $DC \parallel EF$;

hence, $AB \parallel DC$. Also, § 463, $AD \parallel BC$.

Hence, § 140, $ABCD$ is a parallelogram.

Ex. **809**. The diagonal of cube is equal to the product of its edge and $\sqrt{3}$.

**Proof.** Denote the edge by $a$, and the diagonal by $b$.

Then, Ex. 804, $b^2 = a^2 + a^2 + a^2 = 3a^2$; $b = \sqrt{3a^2} = a\sqrt{3}$.

Ex. **810**. The volume of a regular prism is equal to the product of its lateral area and one half the apothem of the base.

**Proof.** Denote the apothem of the base by Ap.

§§ 583, 543, $V = B \times H$; but, Ex. 803, $H = E$, and, § 389, $B = \frac{1}{2}\text{Ap.} \times P.$;

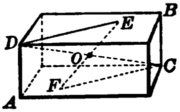
hence, $V = \frac{1}{2}\text{Ap.} \times P \times E$; but, § 520, $A = P \times E$;

consequently, $V = \frac{1}{2}\text{Ap.} \times A$; that is, $V =$ the product of the lateral area and one half the apothem of the base.

Ex. **811**. Any straight line passing through the center of a parallelopiped and terminated by two faces is bisected at the center.

**Data**: Parallelopiped $AB$; its center $O$; and any line, as $EF$, drawn through $O$, and terminated by two faces, as $DB$ and $AC$.

**To prove** $EO = FO$.



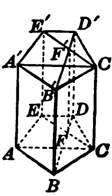
**Proof.** Draw diagonal $DC$, and through $DC$ and $EF$ pass a plane intersecting $DB$ in $DE$ and $AC$ in $FC$.

Then, data, § 529, $DO = CO$, § 59, $\angle DOE = \angle COF$, and since, § 463, $DE \parallel FC$, § 73, $\angle EDO = \angle FCO$; hence, § 102, $\triangle DOE = \triangle COF$, and $EO = FO$.

**Ex. 812.** If any two non-parallel diagonal planes of a prism are perpendicular to the base, the prism is a right prism.

**Data**: Non-parallel diagonal planes as $AC'$ and $EC' \perp$ base $ABCDE$ of the prism $AD'$.

**To prove** $AD'$ a right prism.



**Proof. Case I.** *When the planes intersect in a lateral edge as $CC'$.*

Since the planes $AC'$ and $EC'$ are perpendicular to the base $ABCDE$, § 484, their intersection $CC' \perp ABCDE$; but, § 505, all the lateral edges are parallel to $CC'$; consequently, § 454, all the lateral edges are perpendicular to the base, and, § 507, the prism is a right prism.

**Case II.** *When the planes, as $BD'$ and $EC'$, intersect in a line as $FF'$, which is not a lateral edge.*

As in Case I, it may be shown that $FF' \perp ABCDE$.

§ 463, $BC \parallel B'C'$, $BF \parallel B'F'$, and $CF \parallel C'F'$;

hence, § 469, $\angle CBF = \angle C'B'F'$, and $\angle BCF = \angle B'C'F'$,

and, § 153, $BC = B'C'$; hence, § 102, $\triangle BCF = \triangle B'C'F'$,

and $CF = C'F'$; consequently, § 150, $CC'F'F$ is a ▱,

and $FF' \parallel CC'$; $\therefore$ § 454, $CC' \perp ABCDE$,

and, as in Case I, all the lateral edges are perpendicular to the base, and the prism is a right prism.

**Ex. 813.** The base of a pyramid is 16 sq. ft. and its altitude is 7 ft. What is the area of a section parallel to the base, if it is 2 ft. 6 in. from the apex?

**Solution.** Denote the area of the section by $A$.

Then, § 554, $A : 16 = (2\frac{1}{2})^2 : 7^2$;

whence, $A = \frac{16 \times (2\frac{1}{2})^2}{7^2} = \frac{100}{49} = 2\frac{2}{49}$ sq. ft.

**Ex. 814.** The edges of a rectangular parallelopiped are 3 in., 4 in., and 6 in. What is the area of its diagonal planes and the length of its diagonal line?

**Solution.** Let the base edges be 3 in. and 4 in., and the lateral edge 6 in.

Then, the diagonal of the base $= \sqrt{3^2+4^2} = \sqrt{25} = 5$ in.; consequently, the dimensions of the diagonal plane are 5 and 6, and its area $= 5 \times 6 = 30$ sq. in.

Ex. 804, diag. line of parallelopiped $= \sqrt{3^2 + 4^2 + 6^2} = 7.81+$ in.

Ex. **815**. A portion of a railway embankment is 18 ft. by 380 ft. at the top, and 40 ft. by 380 ft. at the bottom. If its height is 12 ft., how many cubic yards, or loads, of earth does it contain?

**Solution.** This solid may be considered as a prism whose altitude is 380 ft. and base a trapezoid whose parallel sides are 18 ft. and 40 ft. and altitude 12 ft.

Then, §§ 583, 338, $V = B \times H = \frac{1}{2} \times 12\,(18 + 40) \times 380 = 132240$ cu. ft.; that is, $V = 132240 \div 27 = 4897\frac{7}{9}$ cu. yd., or loads.

Ex. **816**. If the four diagonals of a four-sided prism pass through a common point, the prism is a parallelopiped.

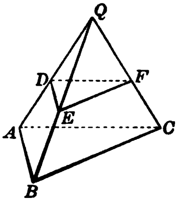
**Data**: (See Fig. for § 529.) Four-sided prism $AG$ whose diagonals $AG$, $BH$, $CE$, and $DF$ pass through a common point $O$.

**To prove** $AG$ a parallelopiped.

**Proof.** Since $AG$ and $DF$ intersect, § 430, they lie in the same plane; $\therefore$ edges $AD$ and $FG$ lie in the same plane; but, § 463, $BC \parallel FG$; $\therefore$ § 457, $BC \parallel$ plane $AFGD$, and consequently cannot meet $AD$; $\therefore$ § 63, $AD \parallel BC$; similarly, $AB \parallel DC$; hence, § 140, $ABCD$ is a parallelogram, and, § 505, $EFGH$ is a parallelogram.

Hence, § 512, $AG$ is a parallelopiped.

Ex. **817**. If a pyramid is cut by a plane parallel to its base, the pyramid cut off is similar to the given pyramid.



**Data**: Pyramid $Q\text{-}ABC$ cut by a plane $DEF \parallel ABC$.

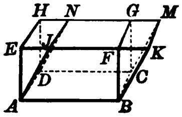
**To prove** $Q\text{-}DEF$ similar to $Q\text{-}ABC$.

**Proof.** § 463, $DE \parallel AB$, $EF \parallel BC$, and $DF \parallel AC$; $\therefore$ § 301, faces $QDE$, $QEF$, and $QDF$ are similar respectively to faces $QAB$, $QBC$, $QAC$, and these faces are similarly placed in each pyramid.

Trihedral $\angle O$ is common to the two pyramids, and, § 500, trihedral ∡ $D$, $E$, and $F$ of $Q\text{-}DEF$ are respectively equal to trihedral ∡ $A$, $B$, and $C$ of $Q\text{-}ABC$.

Hence, § 569, $Q\text{-}DEF$ is similar to $Q\text{-}ABC$.

Ex. **818**. The lateral area of a right prism is less than the lateral area of any oblique prism having the same base and altitude.



**Data**: Right prism $AG$ and oblique prism $AM$ on the same base $ABCD$ and having the same altitude.

**To prove** lat. area of $AG <$ lat. area of $AM$.

**Proof.** § 61, $AE < AJ$; $\therefore AD \times AE < AD \times AJ$; that is, area $ADHE <$ area $ADNJ$; similarly, $BCGF <$ area $BCMK$.

§ 333, area $ABFE =$ area $ABKJ$, and area $DCGH =$ area $DCMN$.

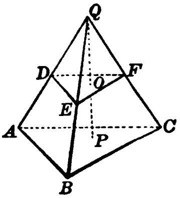
Ax. 4,

$ADHE + BCGF + ABFE + DCGH < ADNJ + BCMK + ABKJ + DCMN$;

that is, lat. area of $AG <$ lat. area of $AM$.

If *AM* were oblique in the other direction also; that is, if it had none of its lateral faces in the same planes with lateral faces of *AG*, much more would the theorem be true; consequently, the lat. area of *AG* is less than that of *any* oblique prism having the same base and altitude.

Ex. **819.** If a section of a pyramid made by a plane parallel to the base bisects the altitude, the area of the section is one fourth the area of the base, and the pyramid cut off is one eighth of the original pyramid.



**Data**: Pyramid $Q$-$ABC$ cut in section $DEF$ by a plane ∥ to its base, and bisecting the altitude $QP$, as at $O$.

**To prove** 1. $DEF = \frac{1}{4} ABC$.

2. Vol. $Q$-$DEF = \frac{1}{8}$ vol. $Q$-$ABC$.

**Proof.** 1. § 554, $DEF : ABC = \overline{QO}^2 : \overline{QP}^2$;

but, data, $QO : QP = 1 : 2$; $\therefore \overline{QO}^2 : \overline{QP}^2 : 1 : 4$;

hence, $DEF : ABC = 1 : 4$; whence, $DEF = \frac{1}{4} ABC$.

2. § 561, $Q\text{-}DEF : Q\text{-}ABC = \frac{1}{4} ABC \times \frac{1}{2} QP : ABC \times QP = \frac{1}{8} : 1$;

that is, vol. $Q$-$DEF = \frac{1}{8}$ vol. $Q$-$ABC$.

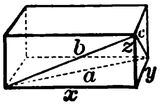
Ex. **820.** The volume of a right triangular prism is equal to one half the product of any lateral face by its distance from the opposite edge.

**Proof.** Denote any edge of the base by $E$, and its distance from the opposite vertex by $D$.

§§ 583, 542, $V = B \times H$; but, § 335, $B = \frac{1}{2} E \times D$;

$\therefore V = \frac{1}{2} E \times D \times H$; that is, $V = \frac{1}{2} D (E \times H)$, $E \times H$ being the area of the lateral face whose base edge is $E$.

Ex. **821.** If the diagonals of three unequal faces of a rectangular parallelopiped are given, compute the edges.



**Solution.** Denote the diagonals of the faces of the parallelopiped by $a$, $b$, and $c$, and its edges by $x$, $y$, and $z$.

§ 349, $x^2 + y^2 = a^2$, (1), and $x^2 + z^2 = b^2$, (2);

then, (1) – (2), $y^2 - z^2 = a^2 - b^2$, (3); but $y^2 + z^2 = c^2$, (4);

$\therefore$ (3) + (4), $2y^2 = a^2 - b^2 + c^2$, and $y = \sqrt{\dfrac{a^2 - b^2 + c^2}{2}}$.

Substituting the value of $y$ in (1), $x^2 + \dfrac{a^2 - b^2 + c^2}{2} = a^2$;

whence, $x^2 = \dfrac{a^2 + b^2 - c^2}{2}$, and $x = \sqrt{\dfrac{a^2 + b^2 - c^2}{2}}$.

Substituting the value of $x$ in (2), $\dfrac{a^2 + b^2 - c^2}{2} + z^2 = b^2$;

whence, $z^2 = \dfrac{b^2 + c^2 - a^2}{2}$, and $z = \sqrt{\dfrac{b^2 + c^2 - a^2}{2}}$.

**Ex. 822.** What is the lateral area of a regular pyramid whose slant height is 10 ft., the base being a pentagon inscribed in a circle whose radius is 6 ft.? What is the volume?

**Solution.** (1) Ex. 713, side of base $= 3\sqrt{10 - 2\sqrt{5}}$;

hence, §§ 583, 551, $A = \frac{1}{2} \times 15\sqrt{10-2\sqrt{5}} \times 10 = 75\sqrt{10-2\sqrt{5}} = 176.25+$ sq. ft.

(2) Ex. 713, apothem of base $= \frac{3}{2}\sqrt{6 + 2\sqrt{5}}$;

hence, § 389, area of base $= \frac{1}{2} \times 15\sqrt{10 - 2\sqrt{5}} \times \frac{3}{2}\sqrt{6 + 2\sqrt{5}} = \frac{45}{2}\sqrt{14.472}+$.

Also, § 350, $H = \sqrt{10^2 - (\frac{3}{2}\sqrt{6 + 2\sqrt{5}})^2} = \sqrt{100 - \frac{9}{4}(6 + 2\sqrt{5})} = \sqrt{76.438}+$.

Hence, §§ 583, 560, $V = \frac{1}{3} \times \frac{45}{2}\sqrt{14.472} \times \sqrt{76.438} = \frac{15}{2}\sqrt{14.472 \times 76.438}$
$= 249.44775+$ cu. ft.

**Ex. 823.** The volume of a rectangular parallelopiped is $336^{\text{cu m}}$, its total area is $320^{\text{sq m}}$, and its altitude is $4^{\text{m}}$. What are the dimensions of its base?

**Solution.** §§ 583, 560, $V = B \times 4 = 336$; whence, $B = 84^{\text{sq m}}$, and the area of both bases $= 84 \times 2 = 168^{\text{sq m}}$;

hence, lateral area $= 320^{\text{sq m}} - 168^{\text{sq m}} = 152^{\text{sq m}}$;

that is, §§ 583, 520, $A = 4 \times P = 152$; whence, $P = 38^{\text{m}}$, and $\frac{1}{2} P = 19^{\text{m}}$.

Consequently, the sum of the dimensions of the base is 19 and their product is 84.

Let $x$ = length of base; then, $19 - x$ = width of base,

and $x(19 - x) = 84$; whence, $x = 12^{\text{m}}$, and $19 - x = 7^{\text{m}}$.

**Ex. 824.** A pyramid weighs $30^{\text{Kg}}$, and its altitude is $12^{\text{dm}}$. A plane parallel to the base cuts off a frustum which weighs $15^{\text{Kg}}$. What is the altitude of the frustum?

**Solution.** Ex. 817, the small pyramid above the frustum is similar to the original pyramid, and weighs $30^{\text{Kg}} - 15^{\text{Kg}} = 15^{\text{Kg}}$.

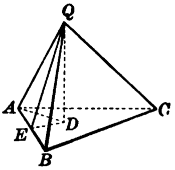
Since the ratio of the weights is the ratio of the volumes, denoting the altitude of the small pyramid by $H$,

§§ 576, 574, $30 : 15 = 12^3 : H^3$;

whence, $H^3 = \dfrac{15 \times 12^3}{30} = 864$, and $H = \sqrt[3]{864} = 9.524+^{\text{ dm}}$;

consequently, the altitude of the frustum $= 12^{\text{dm}} - 9.524+^{\text{ dm}} = 2.476-^{\text{dm}}$.

**Ex. 825.** Each side of the base of a regular triangular pyramid is 3 in., and its altitude is 8 in. What are its lateral edge and lateral area?



**Solution.** Ex. 649, $AB = r\sqrt{3}$; that is, $3 = r\sqrt{3}$;

whence, $r = AD = \sqrt{3}$;

hence, lateral edge,

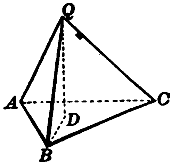
$AQ = \sqrt{\overline{AD}^2 + \overline{QD}^2} = \sqrt{(\sqrt{3})^2 + 8^2} = \sqrt{67} = 8.185+$ in.

Slant height,

$$QE = \sqrt{\overline{AQ}^2 - \overline{AE}^2} = \sqrt{(\sqrt{67})^2 - (\tfrac{3}{2})^2} = \sqrt{67 - \tfrac{9}{4}} = \tfrac{1}{2}\sqrt{259} = 8.0467+ \text{ in.}$$

Then, §§ 583, 551, $A = \frac{1}{2} P \times L = \frac{1}{2} \times 9 \times 8.0467+ = 36.21+$ sq. in.

Ex. **826.** The volume of a regular tetrahedron is equal to the product of the cube of its edge and $\frac{1}{12}\sqrt{2}$.



**Data:** Regular tetrahedron $Q\text{-}ABC$ whose edge is $AB$.

**To prove** vol. $Q\text{-}ABC = \frac{1}{12}\overline{AB}^3\sqrt{2}$.

**Proof.** § 560, vol. $Q\text{-}ABC = \frac{1}{3} ABC \times QD$; (1)

but, Ex. 632,

$$\text{area } ABC = \sqrt{\frac{3\,AB}{2} \times \frac{AB}{2} \times \frac{AB}{2} \times \frac{AB}{2}} = \tfrac{1}{4}\overline{AB}^2\sqrt{3},$$

and, § 350, $QD = \sqrt{\overline{QB}^2 - \overline{BD}^2}$; and since $QB = AB$,

and, Ex. 649, $BD = \dfrac{AB}{\sqrt{3}}$, $QD = \sqrt{\overline{AB}^2 - \dfrac{\overline{AB}^2}{3}} = AB\sqrt{\tfrac{2}{3}}$;

∴ substituting in (1), vol. $Q\text{-}ABC = \frac{1}{3} \times \frac{1}{4}\overline{AB}^2\sqrt{3} \times AB\sqrt{\frac{2}{3}} = \frac{1}{12}\overline{AB}^3\sqrt{2}$.

Ex. **827.** The volume of a regular octahedron is equal to the product of the cube of its edge and $\frac{1}{3}\sqrt{2}$.

**Data:** (See Fig. for § 581, 3.) Regular octahedron $E\text{-}ABCD\text{-}F$ whose edge is $AB$.

**To prove** vol. $E\text{-}ABCD\text{-}F = \frac{1}{3}\overline{AB}^3\sqrt{2}$.

**Proof.** Since $EF \perp ABCD$ and is bisected at $O$,

§ 560, vol. $E\text{-}ABCD\text{-}F = 2 \times \frac{1}{3}\overline{AB}^2 \times OE$. (1)

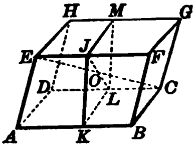
Now, § 350, $OE = \sqrt{\overline{BE}^2 - \overline{OB}^2}$; but $BE = AB$,

and, Ex. 650, $OB = \dfrac{AB}{\sqrt{2}}$; hence, $OE = \sqrt{\overline{AB}^2 - \dfrac{\overline{AB}^2}{2}} = \tfrac{1}{2}AB\sqrt{2}$;

∴ substituting in (1), vol. $E\text{-}ABCD\text{-}F = 2 \times \frac{1}{3}\overline{AB}^2 \times \frac{1}{2}AB\sqrt{2} = \frac{1}{3}\overline{AB}^3\sqrt{2}$.

Ex. **828.** Any plane passing through the center of a parallelopiped divides it into two equal solids.

**Data:** Parallelopiped $AG$ and any plane, as $JKLM$, through its center $O$, dividing $AG$ into the two solids $H\text{-}AKLD$ and $B\text{-}GMJF$.



**To prove** $H\text{-}AKLD = B\text{-}GMJF$.

**Proof.** Draw $EC$ and pass the plane $DCFE$.

Then, this plane intersects $JKLM$ in $JL$, and $JL$ and $EC$ intersect in some point.

Now, § 529, $O$ is the middle point of $EC$, and since, data, $O$ is in $JKLM$, it must be in $JL$; consequently, $EC$ and $JL$ intersect in $O$, and, Ex. 811, $EC$ and $JL$ are bisected at $O$.

Then, in $\triangle EOJ$ and $COL$, $EO = CO$, $JO = LO$,

and, § 59, $\angle EOJ = \angle COL$;

hence, $\triangle EOJ = \triangle COL$, and $EJ = CL$;

similarly, $AK = GM$.

Apply the figure *AEJK* to the figure *GCLM*, so that *AE* shall coincide with its equal *GC*, *A* falling upon *G*, and *E* upon *C*.

Since, §§ 149, 469, $\angle EAK = \angle BFJ = \angle CGM$,

*AK* will coincide with its equal *GM*; similarly, *EJ* will coincide with *CL*, and, Ax. 11, *JK* will coincide with *LM*; hence, the figures coincide in all their parts and are equal.

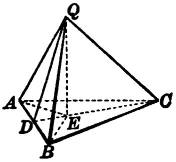
Similarly, $AKLD = GMJF$;

also, § 528, $ADHE = GFBC$.

Hence, § 525, $H\text{-}AKLD = B\text{-}GMJF$.

**Ex. 829.** The lateral area of a regular pyramid is greater than its base.

**Data:** Any regular pyramid, as *Q–ABC*, whose base is *ABC*.



**To prove** lat. area $Q\text{-}ABC > ABC$.

**Proof.** Draw $QE \perp$ plane *ABC* and $ED \perp AB$; also draw *QD*, *AE*, *BE*, and *CE*.

Then, § 451, $QD \perp AB$,

and, § 335, area $ABQ = \frac{1}{2} AB \times QD$;

also, area $ABE = \frac{1}{2} AB \times ED$;

but since, § 448, $QE \perp ED$, $\angle QED$ is a rt. $\angle$,

and, § 128, $QD > ED$;

hence, $\triangle ABQ > \triangle ABE$;

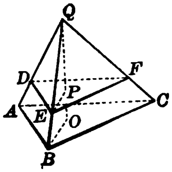
similarly, $\triangle BCQ > \triangle BCE$, etc.;

$\therefore$ Ax. 4, $\triangle ABQ + \triangle BCQ + \text{etc.} > \triangle ABE + \triangle BCE + \text{etc.}$;

that is, lat. area $Q\text{-}ABC > ABC$.

**Ex. 830.** The lateral edge of the frustum of a regular triangular pyramid is $4\frac{1}{2}$ ft., a side of one base is 5 ft., and of the other 4 ft. What is the volume?

**Solution.** Let *DEF–ABC* be the frustum of the regular triangular pyramid *Q–ABC*, in which $BE = 4\frac{1}{2}$ ft., $AB = 5$ ft., and $DE = 4$ ft.; let *QO* be the altitude of the pyramid piercing *DEF* as at *P*.



§ 299, $QB : QE = AB : DE$;

that is, $QE + 4\frac{1}{2} : QE = 5 : 4$;

whence, $QE = 18$.

Ex. 649, $DE = r\sqrt{3} = 4;$

whence, $r^2 = \frac{16}{3} = \overline{PE}^2;$

§ 350, $QP = \sqrt{\overline{QE}^2 - \overline{PE}^2} = \sqrt{18^2 - \frac{16}{3}} = \sqrt{324 - \frac{16}{3}} = \sqrt{\frac{956}{3}} = 17.851+$ ft.

§ 289, $QE : BE = QP : PO;$

that is, $18 : 4.5 = 17.851+ : PO;$

whence, $PO = H = \dfrac{4.5 \times 17.851+}{18} = 4.4627+$ ft.

Ex. 632, area $\triangle ABC = \sqrt{\frac{15}{2} \times \frac{5}{2} \times \frac{5}{2} \times \frac{5}{2}} = \frac{25}{4}\sqrt{3},$

and area $\triangle DEF = \sqrt{6 \times 2 \times 2 \times 2} = 4\sqrt{3}.$

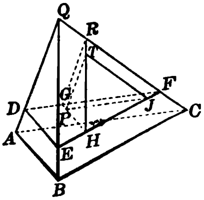
§§ 583, 564, $V = \frac{1}{3}H(B + b + \sqrt{B \times b});$

that is, vol. $DEF\text{-}ABC = \frac{1}{3} \times 4.4627\left(\frac{25}{4}\sqrt{3} + 4\sqrt{3} + \sqrt{\frac{25}{4}\sqrt{3} \times 4\sqrt{3}}\right)$

$= \frac{1}{3} \times 4.4627(\frac{25}{4}\sqrt{3} + 4\sqrt{3} + 5\sqrt{3}) = \frac{1}{3} \times 4.4627 \times 15\frac{1}{4}\sqrt{3}$

$= 39.29+$ cu. ft.

**Ex. 831.** The sum of the perpendiculars from any point within a regular tetrahedron to each of its four faces is equal to its altitude.



**Data**: Regular tetrahedron $Q\text{-}ABC$, and any point $P$ within it.

**To prove** the sum of the ⟂s from $P$ to the faces = the altitude of $Q\text{-}ABC$.

**Proof.** Through $P$ pass plane $DEF \parallel ABC$.

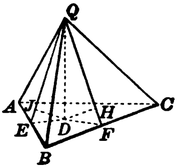
Then, Ex. 817, $Q\text{-}DEF$ is a regular tetrahedron, and alt. of $Q\text{-}ABC$ = alt. of $Q\text{-}DEF$ + the ⟂ from $P$ to $ABC$.

Through $P$ pass plane $GHR \parallel ABQ$; then, $R\text{-}GHF$, or $F\text{-}GHR$ is a regular tetrahedron, and alt. of $Q\text{-}DEF$, or $F\text{-}DEQ$ = alt. of $F\text{-}GHR$ + the ⟂ from $P$ to $ABQ$.

Through $P$ pass plane $PJT \parallel ACQ$; then, $H\text{-}PJT$ is a regular tetrahedron, and alt. of $F\text{-}GHR$, or $H\text{-}GFR$ = alt. of $H\text{-}PJT$ + the ⟂ from $P$ to $ACQ$, and alt. of $H\text{-}PJT$, or $P\text{-}HJT$ = the ⟂ from $P$ to $BCQ$.

Hence, the sum of the ⟂s from $P$ to the faces = the alt. of $Q\text{-}ABC$.

**Ex. 832.** In a regular tetrahedron an altitude is equal to three times the perpendicular from its foot on a face.



**Data**: Regular tetrahedron $Q\text{-}ABC$; its altitude $QD$; and the ⟂ $DH$ from $D$ to a face, as $BQC$.

**To prove** $QD = 3\,DH$.

**Proof.** If it can be proved that $DH$ = each of the other two ⟂s from $D$ to the faces, the proof will rest upon Ex. 831. Pass a plane through $QD$ and $DH$ intersecting $ABC$ as in $DF$ and $BCQ$ as in $FQ$.

Then, § 482, this plane is $\perp$ $ABC$ and $BCQ$; consequently, § 485, it is $\perp$ $BC$; hence, § 443, $DF$ and $FQ \perp BC$.

Also, $QD \perp DF$, and $DH \perp FQ$.

Then, in similar rt. $\triangle$ $DFH$ and $DFQ$,

$$DF : FQ = DH : QD; \text{ whence, } DH = \frac{DF \times QD}{FQ}.$$

Similarly, $$DJ = \frac{DE \times QD}{EQ};$$

but, since $D$ is the center of the $\odot$ circumscribed about $\triangle ABC$, § 202, $DF = DE$; also, $EQ = FQ$; hence, $DH = DJ$; and similarly, $DH =$ the $\perp$ to $ACQ$.

Ex. 831, $QD \doteq$ sum of the $\perp$s from $D$ to the faces of $Q\text{-}ABC$, and since $D$ is in one face, $QD = 3\,DH$.

## BOOK IX

Ex. **833**. What is the lateral area of a cylinder of revolution whose altitude is 18 ft. and the diameter of whose base is 6 ft.?

**Solution.** § 602, $A = 2\pi R \times H = 2 \times 3.1416 \times 3 \times 18 = 339.2928$ sq. ft.

Ex. **834**. What is the volume of a cylinder of revolution whose altitude is 7 ft. and the circumference of whose base is 5 ft.?

**Solution.** § 395, $C = 2\pi R = 5$; whence, $R = \dfrac{5}{2\pi}$;

hence, § 606, $V = \pi R^2 \times H = \pi\left(\dfrac{5}{2\pi}\right)^2 \times 7 = \dfrac{25 \times 7}{4 \times 3.1416} = 13.926+$ cu. ft.

Ex. **835**. How many cubic feet are there in a cylindrical log 14 ft. long and 2.5 ft. in diameter?

**Solution.** § 606, $V = \pi R^2 \times H = 3.1416 \times 1.25^2 \times 14 = 68.7225$ cu. ft.

Ex. **836**. The altitude of a cylinder of revolution is $16^{\text{dm}}$ and the diameter of its base is $11^{\text{dm}}$. What is its total area? What is its volume?

**Solution.**

(1) § 602, $T = 2\pi R(H + R) = 2 \times 3.1416 \times 5.5\,(16 + 5.5) = 742.9884^{\text{sq dm}}$.

(2) § 606, $V = \pi R^2 \times H = 3.1416 \times 5.5^2 \times 16 = 1520.5344^{\text{cu dm}}$.

Ex. **837**. The total area of a cylinder of revolution is $659^{\text{sq dm}}$ and its altitude is $15^{\text{dm}}$. What is the diameter of its base?

**Solution.** § 602, $T = 2\pi R(H + R) = 2 \times 3.1416 \times R(15 + R) = 659$;

whence, $R^2 + 15\,R = 104.8828+$, and $R = 5.193+^{\text{dm}}$;

hence, diameter $= 2\,R = 10.386+^{\text{dm}}$.

Ex. **838**. What is the lateral area of a cone of revolution whose slant height is 13 ft. and the diameter of whose base is 5 ft.?

**Solution.** § 624, $A = \pi RL = 3.1416 \times 2.5 \times 13 = 102.102$ sq. ft.

Ex. **839**. What is the ratio of the lateral surface of a right circular cylinder to that of a right circular cone having the same base and altitude, if the altitude is 5 times the radius of the base?

**Solution.** Denote the lateral area of the cylinder by $A$ and of the cone by $A'$.

Then, § 602, $A = 2\pi R \times 5R = 10\pi R^2$,

and, § 624, $A' = \pi RL$; but, § 349, $L = \sqrt{R^2 + (5R)^2} = R\sqrt{26}$;

hence, $A' = \pi R^2\sqrt{26}$;

consequently, $A : A' = 10\pi R^2 : \pi R^2\sqrt{26} = 10 : \sqrt{26}$.

Ex. **840**. The slant height of a right circular cone is 21 ft. and its altitude is 15 ft. What is its total area?

**Solution.** § 350, $R = \sqrt{21^2 - 15^2} = 14.6969+$ ft.,

then, § 624,

$$T = \pi R(L + R) = 3.1416 \times 14.6969(21 + 14.6969) = 1648.18945+ \text{ cu. ft.}$$

Ex. **841**. The slant height of a right circular cone is $6^{\text{m}}$ and the radius of its base is $5^{\text{m}}$. What is its lateral area? What is its volume?

**Solution.** (1) § 624, $A = \pi RL = 3.1416 \times 5 \times 6 = 94.248^{\text{sq m}}$.

(2) § 350, $H = \sqrt{6^2 - 5^2} = \sqrt{11}$;

then, § 630, $V = \frac{1}{3}\pi R^2 \times H = \frac{1}{3}\pi \times 5^2 \times \sqrt{11} = 86.829+^{\text{cu m}}$.

Ex. **842**. What is the volume of a cone whose altitude is 13 ft. and the circumference of whose base is 9 ft.?

**Solution.** § 395, $C = 2\pi R = 9$; whence, $R = \dfrac{9}{2\pi}$;

then, § 630, $V = \frac{1}{3}\pi R^2 \times H = \frac{1}{3}\pi\left(\dfrac{9}{2\pi}\right)^2 \times 13 = 27.93+$ cu. ft.

Ex. **843**. What is the total area of the frustum of a cone of revolution whose slant height is $17^{\text{dm}}$ and the radii of whose bases are $5^{\text{dm}}$ and $3^{\text{dm}}$?

**Solution.** §§ 635, 627,

lateral area $= \frac{1}{2} \times L(C + C') = \frac{1}{2} \times 17(10\pi + 6\pi) = 8\pi \times 17 = 427.2576^{\text{sq dm}}$

Area of upper base $= \pi\, 3^2 = 28.2744^{\text{sq dm}}$,

and area of lower base $= \pi\, 5^2 = 78.54^{\text{sq dm}}$.

Hence, total area $= 427.2576^{\text{sq dm}} + 28.2744^{\text{sq dm}} + 78.54^{\text{sq dm}} = 534.072^{\text{sq dm}}$.

Ex. **844**. How far from the base must a cone of revolution whose altitude is 15 in. be cut by a plane parallel to the base so that the volume of the frustum shall be one half that of the entire cone?

**Solution.** (See Fig. for § 621.) Let $AOQ$ be the generating $\triangle$ of the cone of revolution, $Q\text{-}ABC$, and let $DEF$ be the section made by the given cutting plane; also let the plane of $\triangle AOQ$ intersect $DEF$ in $DP$.

Then, § 463, $DP \parallel AO$, and, § 301, $\triangle DPQ$ and $AOQ$ are similar; but, they are the generating $\triangle$ of $Q$-$DEF$ and $Q$-$ABC$; hence, § 614, $Q$-$DEF$ and $Q$-$ABC$ are similar.

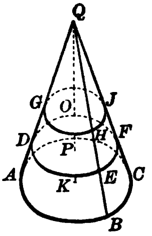
Then, § 631, vol. $Q$-$DEF$ : vol. $Q$-$ABC = \frac{1}{2} : 1 = QP^3 : QO^3$;

that is, $\frac{1}{2} : 1 = \overline{QP}^3 : 15^3$; whence, $\overline{QP}^3 = \frac{1}{2} \times 15^3 = 1687.5$,

and $QP = \sqrt[3]{1687.5} = 11.9+$ in.

Hence, $PO = QO - QP = 15$ in. $- 11.9+$ in. $= 3.1-$ in., the distance of the cutting plane from the base.

**Ex. 845.** At what distances from the vertex must a cone of revolution be cut by planes parallel to the base to divide it into three equivalent solids?



**Solution.** Since the three parts are to be equivalent, cone $Q$-$DEF$ will be equivalent to $\frac{2}{3}$ of cone $Q$-$ABC$, and cone $Q$-$GHJ$ to $\frac{1}{3}$ of cone $Q$-$ABC$.

As in Ex. 844, these cones are similar cones of revolution.

Then, § 631, vol. $Q$-$DEF$ : vol. $Q$-$ABC = \overline{QP}^3 : \overline{QK}^3$;

that is, $2 : 3 = \overline{QP}^3 : \overline{QK}^3$;

whence, $\overline{QP}^3 = \frac{2}{3}\,\overline{QK}^3$, and $QP = QK\sqrt[3]{\frac{2}{3}}$.

Also, vol. $Q$-$GHJ$ : vol. $Q$-$ABC = \overline{QO}^3 : \overline{QK}^3$;

that is, $1 : 3 = \overline{QO}^3 : \overline{QK}^3$;

whence, $\overline{QO}^3 = \frac{1}{3}\,\overline{QK}^3$, and $QO = QK\,\sqrt[3]{\frac{1}{3}}$.

Consequently, the distances of the cutting planes from the vertex are respectively $\sqrt[3]{\frac{1}{3}}$, and $\sqrt[3]{\frac{2}{3}}$ times the altitude of the given cone.

**Ex. 846.** A plane parallel to the base of a cone of revolution cuts the altitude at a point $\frac{2}{3}$ of the distance from the vertex. What is the ratio of the volume of the cone cut off to that of the original cone?

**Solution.** As in Ex. 844, the cones are similar, and their altitudes are in the ratio of $2 : 3$.

Then, § 631, $V : V' = 2^3 : 3^3 = 8 : 27$.

**Ex. 847.** What is the volume of the frustum of a cone whose altitude is 21 ft., and the circumferences of whose bases are 17 ft. and 13 ft. respectively?

**Solution.** § 395, $R = \frac{17}{2\pi}$, and $R' = \frac{13}{2\pi}$;

hence, § 684,

$$V = \tfrac{1}{3}\pi \times 21\left\{\left(\frac{17}{2\pi}\right)^2 + \left(\frac{13}{2\pi}\right)^2 + \frac{17}{2\pi} \times \frac{13}{2\pi}\right\}$$

$$= 7\pi \times \left(\frac{289}{4\pi^2} + \frac{169}{4\pi^2} + \frac{221}{4\pi^2}\right) = 378.23 + \text{cu. ft.}$$

**Ex. 848.** What is the lateral area, total area, and volume of a cylinder of revolution whose diameter is 8 in. and altitude 12 in.?

**Solution.**

1. § 602, $A = 2\pi RH = 2 \times 3.1416 \times 4 \times 12 = 301.5936$ sq. in.

2. $T = 2\pi R(H + R) = 2 \times 3.1416 \times 4 \times 16 = 402.1248$ sq. in.

3. § 606, $V = \pi R^2H = 3.1416 \times 4^2 \times 12 = 603.1872$ cu. in.

**Ex. 849.** What is the lateral area, total area, and volume of a cone of revolution whose base is $10^{\text{cm}}$ in diameter and whose altitude is $12^{\text{cm}}$?

**Solution.** 1. § 349, $L = \sqrt{5^2 + 12^2} = 13^{\text{cm}}$;

hence, § 624, $A = \pi RL = 3.1416 \times 5 \times 13 = 204.204^{\text{sq cm}}$.

2. $T = \pi R(L + R) = 3.1416 \times 5 \times 18 = 282.744^{\text{sq cm}}$.

3. § 630, $V = \frac{1}{3}\pi R^2H = \frac{1}{3} \times 3.1416 \times 5^2 \times 12 = 314.16^{\text{cu cm}}$.

**Ex. 850.** What is the lateral area, total area, and volume of a frustum of a cone of revolution, the radii of whose bases are 6 in. and 4 in. respectively, and whose altitude is 9 in.?

**Solution.** 1. (See Fig. for § 621.) Let $DEF\text{-}ABC$ be the frustum of the cone $Q\text{-}ABC$, in which $AO = 6$ in., $DP = 4$ in., and $OP = 9$ in.

§ 299, $QO : QP = AO : DP$; that is, $QP + 9 : QP = 6 : 4$;

whence, $QP = 18$ in.

§ 349, $QD = \sqrt{\overline{DP}^2 + \overline{QP}^2} = \sqrt{4^2 + 18^2} = \sqrt{340} = 2\sqrt{85}$;

then, § 289, $AD : OP = QD : QP$; that is, $AD : 9 = 2\sqrt{85} : 18$;

whence,
$$AD = \frac{9 \times 2\sqrt{85}}{18} = \sqrt{85}.$$

§§ 635, 627,

$A = \frac{1}{2} L(C + C') = \frac{1}{2}\sqrt{85}(2\pi 6 + 2\pi 4) = 10\pi\sqrt{85} = 289.6398+$ sq. in.

2. § 398, area of lower base $= \pi 6^2 = 3.1416 \times 6^2 = 113.0976$ sq. in.,

and area of upper base $= \pi 4^2 = 3.1416 \times 4^2 = 50.2656$ sq. in.,

then,

total area $= 289.6398+$ sq. in. $+ 113.0976$ sq. in. $+ 50.2656$ sq. in. $= 453.003+$ sq. in.

3. § 634,
$$\begin{aligned} V &= \tfrac{1}{3}\pi H(R^2 + R'^2 + RR') \\ &= \tfrac{1}{3} \times 3.1416 \times 9(6^2 + 4^2 + 4 \times 6) \\ &= \tfrac{1}{3} \times 3.1416 \times 9 \times 76 = 716.2848 \text{ cu. in.} \end{aligned}$$

**Ex. 851.** On a cylindrical surface, only one straight line can be drawn through a given point.

**Proof.** If another straight line besides the element which passes through the given point could be drawn on the cylindrical surface, that surface would coincide with the plane determined by those two intersecting lines, which is impossible.

**Ex. 852.** The intersection of two planes tangent to a cylinder is parallel to an element.

**Proof.** Since the elements of contact are parallel, § 457, each is parallel to the tangent plane through the other; hence, § 458, the intersection of the planes is parallel to either of the elements of contact, or to any element.

Ex. **853**. How many square yards of canvas are required for a conical tent 18 ft. high and 10 ft. in diameter?

**Solution.** § 349, $L = \sqrt{5^2 + 18^2} \doteq 18.68+$ ft.;
then, § 624, $A = \pi RL = 3.1416 \times 5 \times 18.68+ = 293.4254+$ sq. ft.,
or $A = 293.4254+ \div 9 = 32.6028+$ sq. yd.

Ex. **854**. How many cubic feet are there in a piece of round timber 40 ft. long, whose ends are respectively 3 ft. and 1 ft. in diameter?

**Solution.** § 634,
$$\begin{aligned} V &= \tfrac{1}{3}\pi H(R^2 + R'^2 + RR') \\ &= \tfrac{1}{3} \times 3.1416 \times 40\{(1\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + 1\tfrac{1}{2} \times \tfrac{1}{2}\} \\ &= \tfrac{1}{3} \times 3.1416 \times 40 \times \tfrac{13}{4} = 136.136 \text{ cu. ft.} \end{aligned}$$

**Ex. 855**. A cylindrical vessel is $40^{\text{dm}}$ long and $20^{\text{dm}}$ in diameter. What is the weight, in grams, of the water that it will hold?

**Solution.**
§ 606, $V = \pi R^2H = 3.1416 \times 10^2 \times 40 = 12566.4^{\text{cu dm}} = 12566400^{\text{cu cm}}$;
hence, the vessel will hold 12566400 grams of water.

Ex. **856**. What is the weight, in grams, of a piece of lead if, when put under water in a cylindrical tank $24^{\text{cm}}$ in diameter, it causes the level of the water to rise $8^{\text{cm}}$, the specific gravity of lead being 11.4?

**Solution.**
§ 606, vol. of lead = vol. of water displaced $= 3.1416 \times 12^2 \times 8 = 3619.1232^{\text{cu cm}}$.

Since $3619.1232^{\text{cu cm}}$ of water weigh 3619.1232 grams, the same volume of lead weighs $3619.1232 \times 11.4$ grams, or 41258.00448 grams.

Ex. **857**. A cylindrical cistern is $6.4^{\text{m}}$ deep and $5.2^{\text{m}}$ in diameter. How long will it take to fill it if $2^{\text{Hl}}$ flow into it per minute?

**Solution.** § 606, $V = \pi R^2H = 3.1416 \times 2.6^2 \times 6.4 = 135.9181824^{\text{cu m}}$ $= 135918.1824^{\text{cu dm}}$; hence, the cistern holds $135918.1824^{\text{l}}$, or $1359.181824^{\text{Hl}}$, and since $2^{\text{Hl}}$ flow in per minute, it will take as many minutes to fill the cistern as 2 is contained times in 1359.181824, or 679.59+ min. = 11 hr. 19.59+ min.

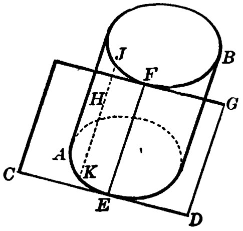
Ex. **858**. A plane, passing through a tangent to the base of a circular cylinder and the element drawn to the point of contact, is tangent to the cylinder.

**Data**: Cylinder $AB$; $CD$ tangent to the base at $E$; the element $EF$; and the plane $CG$ passing through $CD$ and $EF$.

**To prove** $CG$ tangent to the cylinder.

**Proof.** Suppose that $CG$ could meet the cylinder in some other point not in $EF$, as $H$.

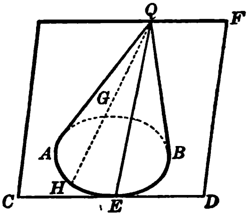
Draw the element $JK$ through $H$.



Then, § 430, *JK* would lie in the plane *CG*, and *CD* would meet the circumference of the base in two points, *E* and *K*; but, Ax. 17, this is impossible.

Hence, the plane *CG* does not meet the cylinder except in the element *EF*; consequently, § 591, *CG* is tangent to the cylinder.

Ex. **859.** A plane, passing through a tangent to the base of a circular cone and the element drawn to the point of contact, is tangent to the cone.



**Data:** Cone *Q-AEB*; *CD* tangent to the base at *E*; the element *QE*; and the plane *CF* passing through *CD* and *QE*.

**To prove** *CF* tangent to the cone.

**Proof.** Suppose that *CF* could meet the cone in some other point not in *QE*, as *G*.

Draw the element *QH* through *G*.

Then, § 430, *QH* would lie in the plane *CF*, and *CD* would meet the circumference of the base in two points, *E* and *H*; but, Ax. 17, this is impossible.

Hence, the plane *CF* does not meet the cylinder except in the element *QE*; consequently, § 615, *CF* is tangent to the cone.

Ex. **860.** The volumes of two similar cylinders of revolution are as 27 : 64. If the diameter of the first is 3 ft., what is the diameter of the second?

**Solution.** Data, $V : V' = 27 : 64$;

but, § 608, $V : V' = 3^3 : D'^3$;

hence, $3^3 : D'^3 = 27 : 64$, and, § 288, $3 : D' = \sqrt[3]{27} : \sqrt[3]{64} = 3 : 4$;

$\therefore$ § 272, $D' = 4$ ft.

Ex. **861.** A cylindrical vessel holds 1728 grams of water. What are the dimensions of the vessel, if the diameter is one third of the altitude?

**Solution.** Since a cubic centimeter of water weighs a gram, the volume of the vessel is $1728^{\text{cu cm}}$.

Then, $V = \pi R^2 H = 1728$; whence, $R^2 H = \dfrac{1728}{\pi}$; (1)

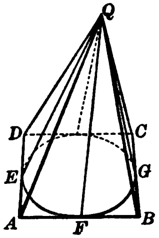
but $D = \frac{1}{3} H$; $\therefore R = \frac{1}{6} H$;

hence, substituting in (1), $(\frac{1}{6} H)^2 H = \dfrac{1728}{\pi}$; that is, $H^3 = \dfrac{1728 \times 36}{3.1416}$;

whence, $H = \sqrt[3]{\dfrac{1728 \times 36}{3.1416}} = 27.054+^{\text{cm}}$,

and $D = \frac{1}{3} H = \frac{1}{3}$ of $27.054+^{\text{cm}} = 9.018+^{\text{cm}}$.

**Ex. 862.** Show that any lateral face of a pyramid circumscribed about a circular cone is tangent to the cone.



**Data:** Pyramid *Q-ABCD* circumscribed about the cone *Q-EFG*.

**To prove** any face of *Q-ABCD*, as *QAB*, tangent to *Q-EFG*.

**Proof.** § 619, *AB* is tangent to the base of the cone.

Draw the element *QF* to the point of contact.

Since *Q* and *F* are in plane *QAB*, § 427, *QF* lies in plane *QAB*; hence, Ex. 859, *QAB* is tangent to the cone.

**Ex. 863.** What is the height of a cylinder $4.8^{dm}$ in diameter, if it is equivalent to a cone of revolution $5.6^{dm}$ in diameter and $6.4^{dm}$ high?

**Solution.** Denote the altitude of the cylinder by $H$.

§ 606, vol. of cylinder $= \pi \times 2.4^2 \times H$,

and, § 630, vol. of cone $= \frac{1}{3}\pi \times 2.8^2 \times 6.4$;

hence, data, $\pi \times 2.4^2 \times H = \frac{1}{3}\pi \times 2.8^2 \times 6.4$;

whence, $H = \dfrac{\pi \times 2.8^2 \times 6.4}{3 \times \pi \times 2.4^2} = 2.9+^{dm}$.

**Ex. 864.** A cylindrical vessel is $12^{cm}$ in diameter and $20^{cm}$ high. How many grams of mercury would it hold, the specific gravity of mercury being 13.6?

**Solution.** § 606, $V = \pi R^2 H = 3.1416 \times 6^2 \times 20 = 2261.962^{cu\,cm}$; consequently, the vessel would hold 2261.962 grams of water,

or $2261.962^g \times 13.6 = 30762.5472$ grams of mercury.

**Ex. 865.** The volumes of similar cones of revolution are to each other as 512 : 729. What is the ratio of their lateral areas?

**Solution.** § 631, $V : V' = R^3 : R'^3$; but, data, $V : V' = 512 : 729$; hence, $R^3 : R'^3 = 512 : 729$, and, § 288, $R : R' = \sqrt[3]{512} : \sqrt[3]{729} = 8 : 9$.

Then, § 625, $A : A' = R^2 : R'^2 = 8^2 : 9^2 = 64 : 81$.

**Ex. 866.** How many centigrams of alcohol will a cylindrical bottle hold, if the bottle is $8^{cm}$ in diameter and $24^{cm}$ high, the specific gravity of alcohol being .79?

**Solution.** § 602, $V = \pi R^2 H = 3.1416 \times 4^2 \times 24 = 1206.3744^{cu\,cm}$; consequently, the bottle would hold 1206.3744 grams of water,

or $1206.3744^g \times .79 = 953.035776^g = 95303.5776^{cg}$ of alcohol.

**Ex. 867.** The specific gravity of marble is 2.8. What is the weight in kilograms, of a conical piece of marble, if the radius of its base is $20^{cm}$ and its height $50^{cm}$?

**Solution.** § 630, $V = \frac{1}{3}\pi R^2 H = \frac{1}{3} \times 3.1416 \times 20^2 \times 50 = 20944^{cu\,cm}$.

The weight of an equal volume of water $= 20944^g$;

hence, the weight of the marble $= 20944^g \times 2.8 = 58643.2^g = 58.6432^{Kg}$

**Ex. 868.** The slant height of a cone of revolution is $3^{m}$. How far from the vertex must the elements be cut by a plane parallel to the base in order that the lateral surface may be divided into two equivalent parts?

**Solution.** As in Ex. 844, the cone cut off is similar to the original cone.

Then, § 626, $A : A' = L^2 : L'^2$;

that is, data, $1 : \frac{1}{2} = 3^2 : L'^2$;

whence, $L'^2 = \frac{9}{2}$, and $L' = \frac{3}{2}\sqrt{2} = 2.121319+ {}^{m}$;

that is, the elements must be cut at the distance $2.121319+ {}^{m}$ from the vertex.

**Ex. 869.** If the altitude of a cylinder of revolution is equal to the diameter of its base, the volume is equal to the product of its total area by one third of its radius.

**Proof.** Data, $D = H$; $\therefore\ R = \frac{1}{2} H$.

Then, § 606, $V = \pi R^2 H = \pi \frac{H^2}{4} H = \frac{1}{4} \pi H^3$.

$$T \times \tfrac{1}{3} R = 2 \pi R(H + R) \times \tfrac{1}{3} R = 2 \pi \frac{H}{2} \left(H + \frac{H}{2}\right) \times \tfrac{1}{6} H$$

$$= 2 \pi \frac{H}{2} \times \frac{3H}{2} \times \frac{H}{6} = \tfrac{1}{4} \pi H^3.$$

Hence, Ax. 1, $V = T \times \frac{1}{3} R$; that is, the volume of the cylinder is equal to the product of its total area by one third of the radius.

## BOOK X

PROP. VII, page 337.

**Proof.** In the second figure draw $PP'$, the axis of the $\odot ABC$, meeting the plane of $ABC$ as in $O$; draw $OB$, $BP$, and $BP'$.

Then, § 652, $O$ is the center of $\odot ABC$, and since, § 107, $\triangle A'B'C' = \triangle ABC$, § 208, Ax. 14, $OB' = OB$.

Since, § 443, $\angle POB$ is a rt. $\angle$, in the rt. $\triangle$s $POB'$ and $POB$, $OB' = OB$, and, const., $PB' = PB$; $\therefore$ § 123, $\triangle POB' = \triangle POB$, and $\angle OPB' = \angle OPB$.

Since, § 227, $\angle PBP'$ is a rt. $\angle$, in the rt. $\triangle$s $PB'P'$ and $PBP'$, $PB' = PB$, and $\angle P'PB' = \angle P'PB$; $\therefore$ § 114, $\triangle PB'P' = \triangle PBP'$, and $PP'$ of the first figure equals $PP'$ the diameter of the sphere; that is, $PG$ which is $\frac{1}{2}$ of $PP'$ is the required radius.

**Ex. 870.** Equal straight lines whose extremities are in the surface of a sphere are equally distant from the center of the sphere.

**Proof.** Pass great circles through the extremities of the given lines.

Then, data, § 654, these lines are equal chords of equal circles; hence, § 202, they are equally distant from the common center of the circles; that is, from the center of the sphere.

**Ex. 871.** The six planes which bisect at right angles the six edges of a tetrahedron all intersect at the same point.

**Data:** Tetrahedron $A\text{-}BCD$, and the six planes bisecting its edges at rt. ∠s.

**To prove** these planes intersect in the same point.

**Proof.** Since, § 676, a sphere may be circumscribed about any tetrahedron, the center $O$ of a sphere circumscribed about $A\text{-}BCD$ is equidistant from $A$, $B$, $C$, and $D$.

Hence, § 452, $O$ is in the plane that bisects the edge $AB$ at rt. ∠s, and similarly $O$ is in each of the six planes which bisect at rt. ∠s the edges of $A\text{-}BCD$; consequently, these planes intersect at the point $O$.

**Ex. 872.** The sides of a spherical triangle are 65°, 86°, and 98°. What are the angles of its polar triangle?

**Solution.** § 702, $\angle A' = 180° - 65° = 115°$,

$\angle B' = 180° - 86° = 94°$,

and $\angle C' = 180° - 98° = 82°$.

**Ex. 873.** The angles of a spherical triangle are 53°, 77°, and 92°. What are the sides of its polar triangle?

**Solution.** § 702, $53° = 180° - a'$; whence, $a' = 127°$,

$77° = 180° - b'$; whence, $b' = 103°$,

and $92° = 180° - c'$; whence, $c' = 88°$.

**Ex. 874.** The angles of a spherical triangle are 65°, 80°, and 110°. What are the sides of its polar triangle?

**Solution.** § 702, $65° = 180° - a$; whence, $a' = 115°$,

$80° = 180° - b'$; whence, $b' = 100°$,

and $110° = 180° - c'$; whence, $c' = 70°$.

**Ex. 875.** If the sides of a spherical triangle are 50°, 75°, and 110°, what are the angles of its polar triangle?

**Solution.** § 702, $\angle A' = 180° - 50° = 130°$,

$\angle B' = 180° - 75° = 105°$,

and $\angle C' = 180° - 110° = 70°$.

**Ex. 876.** If the sides of a spherical triangle are 54°, 89°, and 103°, what is the spherical excess of its polar triangle?

**Solution.** § 702, the ∠s of the polar △ are as follows:
$\angle A' = 180° - 54° = 126°$, $\angle B' = 180° - 89° = 91°$, and $\angle C' = 180° - 103° = 77°$; hence, § 706, its spherical excess $= (126° + 91° + 77°) - 180° = 114°$.

**Ex. 877.** The angle of a lune is 40°. What part of the surface of the sphere is the lune?

**Solution.** § 732, $L : S = 40° : 360°$;

hence, $L = \frac{40 \times S}{360}$; that is, the lune is $\frac{1}{9}$ of the surface of the sphere.

**Ex. 878.** What is the area of a spherical triangle whose angles are 85°, 120°, and 110°, if the radius of the sphere is $10^{\text{dm}}$?

**Solution.** § 706, $E = (85° + 120° + 110°) - 180° = 135°$;

hence, § 737, area of $\triangle$ : area of surf. of sph. $= 135 : 720 = 3 : 16$;

consequently, area of $\triangle = \frac{3}{16}$ of area of surf. of sph.;

but, § 726, area of surf. of sph. $= 4\pi R^2 = 4 \times 3.1416 \times 10^2 = 1256.64^{\text{sq dm}}$;

hence, the area of the $\triangle = \frac{3}{16}$ of $1256.64^{\text{sq dm}} = 235.62^{\text{sq dm}}$.

**Ex. 879.** The area of the surface of a sphere is 160 sq. in.; the angles of a spherical triangle on this sphere are 93°, 117°, and 132°. What is the area of the triangle?

**Solution.** § 706, $E = (93° + 117° + 132°) - 180° = 162°$;

hence, § 737, area of $\triangle$ : area of surf. of sph. $= 162 : 720 = 9 : 40$;

consequently,

area of $\triangle = \frac{9}{40}$ of area of surf. of sph. $= \frac{9}{40}$ of 160 sq. in. = 36 sq. in.

**Ex. 880.** Two spherical triangles on the same sphere, or on equal spheres, are equivalent, if the perimeters of their polar triangles are equal.

**Proof.** If the perimeters of their polars are equal, then, §§ 701, 54, the sums of the ∠s of the given $\triangle$s are equal, consequently, their spherical excesses are equal; hence, § 735, the $\triangle$s are equivalent.

**Ex. 881.** What is the volume of a sphere whose radius is 9 in.?

**Solution.** § 744, $V = \frac{4}{3}\pi R^3 = \frac{4}{3} \times 3.1416 \times 9^3 = 3053.6352$ cu. in.

**Ex. 882.** The circumference of a great circle of a sphere is 36 ft. What is the area of the surface of the sphere?

**Solution.** § 395, $C = 2\pi R = 36$; whence, $R = \frac{18}{\pi}$;

hence, § 726, $A = 4\pi R^2 = 4 \times \pi \times \left(\frac{18}{\pi}\right)^2 = \frac{4 \times 18^2}{3.1416} = 412.5+$ cu. ft.

**Ex. 883.** The diameter of a sphere is $13^{\text{dm}}$. How many cubic decimeters does it contain?

**Solution.** § 744, $V = \frac{1}{6}\pi D^3 = \frac{1}{6} \times 3.1416 \times 13^3 = 1150.3492^{\text{cu dm}}$.

**Ex. 884.** The volume of a sphere is $1870^{\text{cu m}}$. What is its radius?

**Solution.** § 744, $V = \frac{4}{3}\pi R^3$; whence, $R^3 = \frac{3V}{4\pi}$;

that is, $R^3 = \frac{3 \times 1870}{4 \times 3.1416} = 446.428571+$;

hence, $R = \sqrt[3]{446.428571} = 7.64+^{\text{m}}$.

**Ex. 885.** The area of the surface of a sphere is 69 sq. ft. What is its diameter?

**Solution.** § 726, $A = 4\pi R^2 = 69$; whence, $R^2 = \dfrac{69}{4 \times 3.1416} \doteq 5.4908+$,

and $R = \sqrt{5.4908+} = 2.34+$ ft.;

hence, the diameter = 2 times 2.34+ ft. = 4.68+ ft.

**Ex. 886.** The edge of a cube is $16^{cm}$. What is the volume of the circumscribed sphere?

**Solution.** Since the diameter of the sphere is the diagonal of the cube,

Ex. 809, $D = 16\sqrt{3}$.

Hence, § 744,

vol. of sph. $= \frac{1}{6}\pi D^3 = \frac{1}{6} \times 3.1416 \times (16\sqrt{3})^3 = 11144.004157+^{\text{cu cm}}$.

**Ex. 887.** If the sides of a spherical triangle are 75°, 93°, and 110°, what is the spherical excess of its polar triangle?

**Solution.** § 702, $\angle A' = 180° - 75° = 105°$,

$\angle B' = 180° - 93° = 87°$,

and $\angle C' = 180° - 110° = 70°$;

hence, the spherical excess of the polar $\triangle = 105° + 87° + 70° - 180° = 82°$.

**Ex. 888.** The angles of a spherical triangle are 98°, 110°, and 160°. What is the area of a symmetrical triangle on the same sphere, the radius being $12^m$?

**Solution.** § 726, $A = 4\pi R^2 = 4 \times 3.1416 \times 12^2 = 1809.5616^{\text{sq m}}$,

and, § 706, $E = (98° + 110° + 160°) - 180° = 188°$.

Hence, § 737, area of $\triangle : 1809.5616 = 188 : 720$;

whence, area of $\triangle = \dfrac{1809.5616 \times 188}{720} = 472.49664^{\text{sq m}}$;

consequently, § 709, the area of a symmetrical $\triangle$ is $472.49664^{\text{sq m}}$.

**Ex. 889.** Find the volume of a triangular spherical pyramid, the angles of the base being 58°, 116°, and 145°, and the diameter of the sphere being 20 in.

**Solution.** § 706, $E = (58° + 116° + 145°) - 180° = 139°$;

hence, § 737, area of base $= \frac{139}{720}$ of area of surface of sphere;

but, § 726, $A = 4\pi R^2 = 4 \times \pi \times 10^2$; hence, area of base $= \frac{139}{720} \times 4 \times \pi \times 10^2$.

Then, § 746, $V = B \times \frac{1}{3}R = \frac{139}{720} \times 4 \times 3.1416 \times 10^2 \times \frac{1}{3} \times 10 = 808.67+$ cu. in.

**Ex. 890.** What is the area of a lune whose angle is 60° on the surface of a sphere whose radius is 6 in.?

**Solution.** § 733, $L = 60 \times 2 = 120$ spherical degrees;

hence, the area of the lune $= \frac{120}{720} = \frac{1}{6}$ of the area of the surface of the sphere,

and since, § 726, $A = 4\pi R^2 = 4 \times 3.1416 \times 6^2 = 452.3904$ sq. in.,

and the area of the lune $= \frac{1}{6}$ of 452.3904 sq. in. = 75.3984 sq. in.

Ex. **891**. What is the area of a zone whose altitude is $3^{\text{dm}}$ on the surface of a sphere whose radius is $8^{\text{dm}}$?

**Solution.** §§ 751, 728, $A = 2\pi RH = 2 \times 3.1416 \times 8 \times 3 = 150.7968^{\text{sq dm}}$.

Ex. **892**. What is the volume of a spherical sector whose altitude is $3.5^{\text{m}}$, if the radius of the sphere is $11^{\text{m}}$?

**Solution.** § 748, $V = \frac{2}{3}\pi R^2 H = \frac{2}{3} \times 3.1416 \times 11^2 \times 3.5 = 886.9784^{\text{cu m}}$.

Ex. **893**. What is the volume of a spherical wedge whose angle is 72°, the volume of the sphere being 1728 cu. in.?

**Solution.** Since, § 733, the lune which is the base of the wedge $= 72 \times 2 = 144$ spherical degrees $= \frac{144}{720}$ or $\frac{1}{5}$ of the surface of the sphere, the volume of the wedge $= \frac{1}{5}$ of 1728 cu. in. $= 345\frac{3}{5}$ cu. in.

Ex. **894**. What is the area of a zone of one base, if the chord of its generating arc is $13^{\text{dm}}$?

**Solution.** § 730, area $= \pi\, 13^2 = 3.1416 \times 13^2 = 530.9304^{\text{cu dm}}$.

Ex. **895**. The area of a zone of a sphere $20^{\text{dm}}$ in diameter is $150^{\text{sq dm}}$. What is the altitude of the zone?

**Solution.** §§ 751, 728, $A = 2\pi RH = 2 \times 3.1416 \times 10 \times H = 150$;

whence, $$H = \frac{150}{2 \times 3.1416 \times 10} = 2.387+^{\text{dm}}.$$

Ex. **896**. The angles of the base of a triangular spherical pyramid are 90°, 121°, and 135°. What is the volume of the pyramid, the volume of the sphere being 194 cu. in.?

**Solution.** Since, § 706, $E = (90° + 121° + 135°) - 180° = 166°$,

§ 737, base of pyr. : surf. of sph. $= 166 : 720$;

but vol. of pyr. : vol. of sph. = base of pyr. : surf. of sph.;

hence, vol. of pyr. : $194 = 166 : 720$;

whence, vol. of pyr. $= \dfrac{194 \times 166}{720} = 44.728-$ cu. in.

Ex. **897**. Spherical polygons are to each other as their spherical excesses.

**Proof.** Denote any two spherical polygons by $A$ and $B$ and their spherical excesses by $E$ and $E'$, respectively.

Then, § 736, $A \Bumpeq E$ sph. deg., and $B \Bumpeq E'$ sph. deg.;

hence, $A : B = E$ sph. deg. : $E'$ sph. deg.; that is, $A : B = E : E'$.

Ex. **898**. The base of a spherical pyramid is a trirectangular triangle. What part of the sphere is the pyramid?

**Solution.** Since, § 706, $E = 90° \times 3 - 180° = 90°$,

§ 737, base of pyr. : surf. of sph. $= 90 : 720 = 1 : 8$;

but pyr. : sph. = base of pyr. : surf. of sph.;

hence, pyr. : sph. $= 1 : 8$; that is, pyr. $= \frac{1}{8}$ of sph.

**Ex. 899.** The surface of a sphere is equivalent to the lateral surface of the circumscribing cylinder.

**Proof.** § 726, area of surface of sphere $= 4\,\pi R^2$.

Now, the radius of the cylinder $= R$, and its altitude $= 2\,R$; $\therefore$ § 602, the area of the lateral surface of the cylinder $= 2\,\pi R \times 2\,R = 4\,\pi R^2$; hence, the surface of the sphere is equivalent to the lateral surface of the cylinder.

**Ex. 900.** What is the spherical excess of a triangle whose area is 261.8 sq. in., if the radius of the sphere is 10 in.?

**Solution.** § 726, $A = 4\,\pi R^2 = 4 \times 3.1416 \times 10^2 = 1256.64$ sq. in.

Then, denoting the number which expresses the spherical excess by $E$,

§ 737, $$261.8 : 1256.64 = E : 720;$$

whence, $$E = \frac{261.8 \times 720}{1256.64} = 150;$$

that is, the spherical excess is 150°.

**Ex. 901.** A lune and a trirectangular spherical triangle on the same sphere are to each other as the angle of the lune is to an angle of 45°.

**Proof.** A trirectangular triangle is equivalent to a lune whose angle is 45°, because it may be divided into half lunes having an angle of 45°, and since, § 734, lunes on the same sphere are to each other as their angles,

$$\text{the lune} : \triangle = \angle \text{ of lune} : 45°.$$

**Ex. 902.** Trirectangular triangles on equal spheres are equal.

**Proof.** Since, § 654, great circles of equal spheres are equal, quadrants of their circumferences are equal and the sides of the respective $\triangle$ are equal; hence, § 712, the $\triangle$ are equal or equivalent, and, since they are isosceles, they may be made to coincide, and are equal.

**Ex. 903.** The diameters of two spheres are 12 in. and 14 in., respectively. What is the ratio of their surfaces? What is the ratio of their volumes?

**Solution.** (1) § 727, $A : A' = D^2 : D'^2 = 12^2 : 14^2 = 144 : 196 = 36 : 49$.

(2) § 745, $V : V' = D^3 : D'^3 = 12^3 : 14^3 = 1728 : 2744 = 216 : 343$.

**Ex. 904.** The areas of the surfaces of two spheres are as 144 to 24. What is the ratio of their diameters? What is the ratio of their volumes?

**Solution.** (1) § 727, $A : A' = D^2 : D'^2$; but, data, $A : A' = 144 : 24$; hence, $D^2 : D'^2 = 144 : 24$, and, § 288, $D : D' = \sqrt{144} : \sqrt{24} = 12 : 2\sqrt{6}$.

(2) § 745, $V : V' = D^3 : D'^3 = 12^3 : (2\sqrt{6})^3 = 1728 : 48\sqrt{6} = 36 : \sqrt{6}$.

**Ex. 905.** The diameters of the sun and earth are in the ratio of 109 : 1. What is the ratio of their volumes?

**Solution.** § 745, $V : V' = D^3 : D'^3$; but, data, $D : D' = 109 : 1$;

hence, § 288, $D^3 : D'^3 = 109^3 : 1^3$;

$\therefore$ $V : V' = 109^3 : 1^3 = 1295029 : 1$.

**Ex. 906.** How many quarts of water will a hemispherical kettle hold, if its inside diameter is 12 in.?

**Solution.** § 744, $V = \frac{1}{2}(\frac{4}{3}\pi R^3) = \frac{1}{2} \times \frac{4}{3} \times 3.1416 \times 6^3 = 452.3904$ cu. in.; but a quart contains 57.75 cu. in.; hence, the kettle will hold $452.3904 \div 57.75 = 7.83+$ qt.

**Ex. 907.** If lines are drawn from any point in the surface of a sphere to the ends of a diameter, they are perpendicular to each other.

**Proof.** The section of a sphere made by a plane passing through the lines is a great ⊙ of the sphere, and the ∠ formed by the lines is inscribed in a semicircle; hence, §§ 227, 26, the lines are perpendicular to each other.

**Ex. 908.** What is the circumference of a small circle of a sphere whose diameter is $9^{dm}$, if the circle is at a distance of $3^{dm}$ from the center?

**Solution.** (See Fig. for § 643.) Let $AC$ be the radius of the small ⊙, $OC$ the $\perp$ from the center of the sphere to the plane of the small ⊙, and $OA$ the radius of the sphere.

§ 350, $$AC = \sqrt{\overline{OA}^2 - \overline{OC}^2} = \sqrt{(\tfrac{9}{2})^2 - 3^2} = \sqrt{\frac{81 - 36}{4}} = \tfrac{3}{2}\sqrt{5}.$$

Then, § 395, the circum. of small ⊙ $= 2\pi \times \frac{3}{2}\sqrt{5} = 3\pi\sqrt{5} = 21.075$ nearly.

**Ex. 909.** The dihedral angles of a spherical pyramid are 40°, 80°, and 120°, and its edge is 9 ft. What is the volume of the pyramid?

**Solution.** § 694, the ∠s of the base of the pyramid are 40°, 80°, and 120°;

then, § 706, $$E = (40° + 80° + 120°) - 180° = 60°;$$

hence, § 737, the area of the base $= \frac{60}{720}$ or $\frac{1}{12}$ of the area of the surf. of the sph.

Now, since the edge of the pyramid is the radius of the sphere,

§ 726, $$A = 4\pi R^2 = 4 \times 3.1416 \times 9^2 \text{ sq. ft.};$$

hence, the area of the base of the pyramid $= \frac{1}{12}$ of $4 \times 3.1416 \times 9^2$
$= 27 \times 3.1416 = 84.8232$ sq. ft.;

consequently, § 746, $V = B \times \frac{1}{3}R = 84.8232 \times \frac{1}{3} \times 9 = 254.4696$ cu. ft.

**Ex. 910.** The dihedrals of a trihedral angle whose vertex is at the center of a sphere are 75°, 90°, and 130°. What is the volume of the part of the sphere included by the faces of the trihedral, the radius of the sphere being $8^m$?

**Solution.** § 694, the ∠s of the base of the spherical pyramid included by the faces of the trihedral are 75°, 90°, and 130°;

then, § 706, $$E = (75° + 90° + 130°) - 180° = 115°;$$

hence, § 737, base of pyr. : surf. of sph. = 115 : 720 = 23 : 144;

but vol. of pyr. : vol. of sph. = base of pyr. : surf of sph.;

consequently, vol. of pyr. : vol. of sph. = 23 : 144;

but, § 744, vol. of sph. $= \frac{4}{3}\pi R^3 = \frac{4}{3}\pi 8^3$;
hence, vol. of pyr. : $\frac{4}{3}\pi 8^3 = 23 : 144$;
whence, vol. of pyr. $= \dfrac{4 \times 3.1416 \times 8^3 \times 23}{3 \times 144} = 342.55+^{\text{cu m}}$

Ex. **911**. What is the radius of a sphere which is equivalent to the sum of two spheres whose radii are respectively 4 in. and 7 in.?

**Solution.** Denote the volumes of the given spheres by $V$ and $V'$, the volume of the sphere which is equivalent to their sum by $V''$, and its radius by $R''$.

Then, § 744, $V = \frac{4}{3}\pi 4^3$, $V' = \frac{4}{3}\pi 7^3$, and $V'' = \frac{4}{3}\pi R''^3$;
but, data, $V'' = V + V' = \frac{4}{3}\pi 4^3 + \frac{4}{3}\pi 7^3 = \frac{4}{3}\pi(4^3 + 7^3)$;
hence, $\frac{4}{3}\pi R''^3 = \frac{4}{3}\pi(4^3 + 7^3)$; whence, $R''^3 = 4^3 + 7^3$,
and $R'' = \sqrt[3]{4^3 + 7^3} = \sqrt[3]{64 + 343} = \sqrt[3]{407} = 7.41+$ in.

Ex. **912**. How many cubic decimeters does a segment of a single base contain, if it is cut from a sphere $12^{\text{dm}}$ in diameter, the altitude of the segment being $4^{\text{dm}}$?

**Solution.** Let the segment be that generated by $ADM$ (Fig. for § 749).
§ 750, $V = \frac{1}{2}\pi r^2 H + \frac{1}{6}\pi H^3$;
but $H = 4^{\text{dm}}$, and $r = AD = \sqrt{\overline{OA}^2 - \overline{OD}^2} = \sqrt{\overline{OA}^2 - (OM - DM)^2}$
$= \sqrt{6^2 - (6-4)^2} = \sqrt{6^2 - 2^2} = \sqrt{36 - 4} = \sqrt{32}$;
hence, $V = \frac{1}{2}\pi(\sqrt{32})^2 \times 4 + \frac{1}{6}\pi 4^3 = \pi(64 + \frac{32}{3}) = \frac{1}{3}\pi \times 224$
$= \frac{1}{3} \times 3.1416 \times 224 = 234.5728^{\text{cu dm}}$

Ex. **913**. In a sphere whose diameter is 20 ft., what is the volume of a segment, the bases of which are on the same side of the center, one 3 ft. and the other 5 ft. from it?

**Solution.** For the notation and Fig. see § 749.

$$r = AD = \sqrt{\overline{OA}^2 - \overline{OD}^2} = \sqrt{10^2 - 3^2} = \sqrt{91},$$
$$r' = BC = \sqrt{\overline{OB}^2 - \overline{OC}^2} = \sqrt{10^2 - 5^2} = \sqrt{75},$$
and $H = DC = OC - OD = 5 - 3 = 2$;
substituting these values in § 749, $V = \frac{1}{2}H(\pi r^2 + \pi r'^2) + \frac{1}{6}\pi H^3$.
$V = \frac{1}{2} \times 2\{\pi(\sqrt{91})^2 + \pi(\sqrt{75})^2\} + \frac{1}{6}\pi 2^3 = \pi(91 + 75 + \frac{4}{3}) = \frac{1}{3}\pi \times 502$
$= \frac{1}{3} \times 3.1416 \times 502 = 525.6944$ cu. ft.

Ex. **914**. Find the area of the surface of a sphere inscribed in a cube whose surface is 726 sq. ft.

**Solution.** Area of one face of cube $= 726 \div 6 = 121$ sq. ft.; hence the edge of the cube $= \sqrt{121} = 11$ ft.; that is, the diameter of the sphere $= 11$ ft., and its radius $= 5\frac{1}{2}$ ft.; consequently,
§ 726, area of surf. of sph. $= 4\pi R^2 = 4 \times 3.1416 \times (\frac{11}{2})^2 = 380.1336$ sq. ft.

**Ex. 915.** A trirectangular triangle and a lune on the same sphere are in the ratio of 14 : 9. What is the angle of the lune and what part of the surface of the sphere is the lune?

**Solution.** (1) Denote the angle of the lune by $A$.

Then, Ex. 901, $\triangle : L = 45 : A$; but, data, $\triangle : L = 14 : 9$;

hence, $45 : A = 14 : 9$; whence, $A = \frac{45 \times 9}{14} = 28\tfrac{13}{14}$ degrees.

(2) § 732, $L : S = \frac{45 \times 9}{14} : 360$;

$$\therefore L = \frac{45 \times 9}{360 \times 14} \text{ of } S = \frac{9}{8 \times 14} \text{ of } S = \frac{9}{112} \text{ of } S.$$

**Ex. 916.** Find the area of a spherical quadrilateral whose angles are 120°, 130°, 140°, and 150°, the volume of the sphere being 1000 cu. ft.

**Solution.** § 744, $V = \tfrac{4}{3}\pi R^3 = 1000$; whence, $R^3 = 1000 \times \frac{3}{4\pi}$,

and $R = 10\sqrt[3]{\frac{3}{4\pi}} = 6.203+$ ft.;

hence, $A = 4\pi R^2 = 4 \times 3.1416 \times 6.203^2 = 483.5+$ sq. ft.

§ 707, $E = (120° + 130° + 140° + 150°) - 180° \times 2 = 180°$;

hence, § 737, area of quad. : 483.5 = 180 : 720;

whence, area of quad. $= \frac{483.5 \times 180}{720} = 120.87+$ sq. ft.

**Ex. 917.** The base of a cone of revolution is the great circle of a sphere, and its altitude is the radius of the sphere. What is the ratio of the surface of the sphere to the lateral surface of the cone?

**Solution.** § 726, for sphere, $A = 4\pi R^2$, and for cone, § 624, $A = \pi RL$;

but, § 349, $L = \sqrt{R^2 + R^2} = R\sqrt{2}$; $\therefore A = \pi R^2\sqrt{2}$;

hence, surf. of sph. : lat. surf. of cone $= 4\pi R^2 : \pi R^2\sqrt{2} = 4 : \sqrt{2}$.

**Ex. 918.** The base of a cone is equal to a great circle of a sphere, and its altitude is equal to the diameter of the sphere. What is the ratio of their volumes?

**Solution.** § 630, vol. of cone $= \tfrac{1}{3}\pi R^2 H = \tfrac{1}{3}\pi R^2 \times 2R = \tfrac{2}{3}\pi R^3$,

and, § 744, vol. of sphere $= \tfrac{4}{3}\pi R^3$;

hence, vol. of cone : vol. of sphere $= \tfrac{2}{3}\pi R^3 : \tfrac{4}{3}\pi R^3 = 1 : 2$.

**Ex. 919.** Find the altitude of a zone whose area is equal to that of a great circle of a sphere, the radius of the sphere being $8^{\text{dm}}$.

**Solution.** § 398, area of a gt. $\odot = \pi R^2$, and, § 728, area of zone $= 2\pi RH$;

hence, data, $2\pi RH = \pi R^2$; whence, $H = \frac{R}{2} = \frac{8}{2} = 4^{\text{dm}}$.

Ex. **920**. How many spherical bullets $\frac{1}{2}$ in. in diameter can be molded from a spherical piece of lead $\frac{1}{2}$ ft. in diameter?

**Solution.** § 745, vol. of lead : vol. of 1 bullet $= 6^3 : (\frac{1}{2})^3$; or as $216 : \frac{1}{8}$ or as 1728 : 1, ∴ 1728 bullets can be made.

Ex. **921**. A cannon ball put into a cylindrical tub 2 ft. in diameter causes the water in the tub to rise 2 in. What is the diameter of the cannon ball?

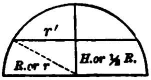
**Solution.** § 606,

vol. of water displaced $= \pi R^2 H = \pi\, 12^2 \times 2 = 288\,\pi$ cu..in.;

hence, § 744, vol. of cannon ball $= \frac{1}{6}\pi D^3 = 288\,\pi$;

whence, $D^3 = 1728$, and $D = \sqrt[3]{1728} = 12$ in.

Ex. **922**. A section parallel to the base of a hemisphere bisects its altitude. What is the ratio of the volumes of the spherical segments thus formed?



**Solution.** Denote the radius of the hemisphere by $R$.

Then, using notation for § 749,

$$r = R,\ r' = \sqrt{R^2 - (\tfrac{1}{2}R)^2} = \tfrac{1}{2}R\sqrt{3},$$

and $H = \frac{1}{2}R$ for the segment having two bases.

Substituting these values in the formula, § 749,

$$V = \frac{1}{2} \times \frac{R}{2}(\pi R^2 + \tfrac{3}{4}\pi R^2) + \tfrac{1}{6}\pi\left(\frac{R}{2}\right)^3 = \tfrac{1}{4}\pi R^3 + \tfrac{3}{16}\pi R^3 + \tfrac{1}{48}\pi R^3 = \tfrac{22}{48}\pi R^3.$$

Now the base of the upper segment is the upper base of the segment of two bases; hence its radius $= \frac{1}{2}R\sqrt{3}$, and the altitude of the segment $= \frac{1}{2}R$. Denote the volume by $V'$.

Substituting these values in the formula, § 750,

$$V' = \tfrac{1}{2}\pi(\tfrac{1}{2}R\sqrt{3})^2\left(\frac{R}{2}\right) + \tfrac{1}{6}\pi\left(\frac{R}{2}\right)^3 = \tfrac{3}{16}\pi R^3 + \tfrac{1}{48}\pi R^3 = \tfrac{10}{48}\pi R^3.$$

Then, $V : V' = \frac{22}{48}\pi R^3 : \frac{10}{48}\pi R^3 = 22 : 10 = 11 : 5.$

Ex. **923**. The volume of a sphere is to that of the circumscribed cube as $\pi$ is to 6.

**Proof.** § 744, vol. of sphere $= \frac{1}{6}\pi D^3$, and since the edge of the cube is $D$, the vol. of the cube $= D^3$;

hence, vol. of sphere : vol. of cube $= \frac{1}{6}\pi D^3 : D^3 = \frac{1}{6}\pi : 1 = \pi : 6$.

Ex. **924**. The volume of a sphere is to that of the inscribed cube as

$$\pi \text{ is to } \frac{2}{\sqrt{3}}.$$

**Proof.** Denote the edge of the cube by $E$;

then, Ex. 809, diag. of cube $= E\sqrt{3}$; but diag. $= 2R$;

hence, $E\sqrt{3} = 2R$, and $E = \dfrac{2R}{\sqrt{3}}$;

consequently, vol. of cube $= \left(\dfrac{2R}{\sqrt{2}}\right)^3 = \dfrac{8R^3}{3\sqrt{3}}$.

§ 744, vol. of sphere $= \frac{4}{3}\pi R^3$;

hence, vol. of sphere : vol. of cube : $\frac{4}{3}\pi R^3 : \dfrac{8R^3}{3\sqrt{3}} = \pi : \dfrac{2}{\sqrt{3}}$.

Ex. **925.** The surface of a sphere is to the total surface of the circumscribing cylinder as 2 is to 3.

**Proof.** § 726, $A = 4\pi R^2$,

and since the radius of the cylinder is $R$ and its altitude $2R$,

§ 602, $T = 2\pi R(H + R) = 2\pi R(2R + R) = 6\pi R^2$;

hence, surf. of sph. : total surf. of cyl. $= 4\pi R^2 : 6\pi R^2 = 2 : 3$.

Ex. **926.** The volume of a sphere is to the volume of a circumscribing cylinder as 2 is to 3.

**Proof.** § 744, vol. of sph. $= \frac{4}{3}\pi R^3$,

and since the radius of the cylinder is $R$ and its altitude $2R$,

§ 606, vol. of cyl. $= \pi R^2 H = 2\pi R^3$;

hence, vol. of sph. : vol. of cyl. $= \frac{4}{3}\pi R^3 : 2\pi R^3 = 2 : 3$.

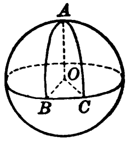
Ex. **927.** A sphere is cut by five parallel planes at the distance of $2^{\text{dm}}$, $3^{\text{dm}}$, $4^{\text{dm}}$, and $5^{\text{dm}}$ from each other, respectively. What are the relative areas of the zones included between the planes?

**Solution.** § 728, areas of zones are respectively, $2\pi R \times 2$, $2\pi R \times 3$, $2\pi R \times 4$, and $2\pi R \times 5$, or $4\pi R$, $6\pi R$, $8\pi R$, $10\pi R$; hence, the zones are to each other as $2 : 3 : 4 : 5$.

Ex. **928.** The sides opposite the equal angles of a birectangular spherical triangle are quadrants.

**Data:** Birectangular sph. $\triangle ABC$ whose rt. $\angle$s are $B$ and $C$.

**To prove** arcs $AB$ and $AC$ are quadrants.



**Proof.** In the corresponding trihedral, dihedrals $BO$ and $CO$ are right dihedrals; that is, planes $AOB$ and $AOC \perp$ plane $BOC$; hence, § 484, $AO \perp$ plane $BOC$; $\therefore$ § 443, $\angle$s $AOB$ and $AOC$ are rt. $\angle$s, and, § 224, arcs $AB$ and $AC$ are quadrants.

Ex. **929.** The slant height of a cone of revolution is equal to the diameter of its base. What is the ratio of its total area to that of the inscribed sphere?

**Solution.** The sphere is that generated by the revolution of the inscribed ⊙ of an equilateral △ whose side is the slant height of the cone; hence, the radius of the sphere is the apothem of that △.

Then, denoting the side of the △ by $2R$, and the radius of its circumscribed ⊙ by $r$,

Ex. 649, $$2R = r\sqrt{3}; \text{ whence, } r = \frac{2R}{\sqrt{3}},$$

and $$\text{the apothem} = \tfrac{1}{2}r = \frac{R}{\sqrt{3}} = \text{the radius of the sphere};$$

hence, § 726, $$\text{area of sphere} = 4\pi\left(\frac{R}{\sqrt{3}}\right)^2 = 4\pi\frac{R^2}{3} = \tfrac{4}{3}\pi R^2,$$

and, § 624, $$T = \pi R(L + R) = \pi R(2R + R) = 3\pi R^2;$$

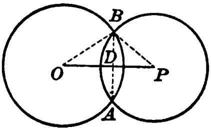
hence, $$T : \text{area of sphere} = 3\pi R^2 : \tfrac{4}{3}\pi R^2 = 9 : 4.$$

**Ex. 930.** The smallest circle whose plane passes through a given point within a sphere is that one whose plane is perpendicular to the radius through the given point.

**Proof.** § 649, the radius to the given point is part of the axis of the ⊙ whose plane is perpendicular to that radius; hence, § 652, the given point is the center of that ⊙.

Then, § 443, any chord of that ⊙ through the given point is perpendicular to the radius, and, Ex. 195, is the shortest chord of any great circle of the sphere that can be drawn through the given point. But this chord is the diameter of the small ⊙; hence, this ⊙ is less than any other whose plane passes through the given point.

**Ex. 931.** The intersection of the surfaces of two spheres is the circumference of a circle whose plane is at right angles to the line joining the centers of the spheres, and whose center is on that line.



**Data:** Two intersecting spheres whose centers are $O$ and $P$, and their line of centers $OP$.

**To prove** the intersection of the spheres a ⊙ whose center is on $OP$, and whose plane is $\perp OP$.

**Proof.** Pass a plane through $O$ and $P$, cutting the spheres in the great Ⓢ $OAB$ and $PAB$ respectively, whose circumferences intersect in $A$ and $B$. Draw $AB$, intersecting $OP$ as at $D$.

Ax. 14, $O$ and $P$ are each equidistant from $A$ and $B$; hence, § 106, $OP \perp AB$ at its middle point $D$.

If the figure is rotated about $OP$ as an axis, the spheres will be generated, and since $DA (= DB)$ is constant in length and perpendicular to $OP$, $DA$ will generate a ⊙ whose center is in $OP$, § 444, whose plane is $\perp OP$, and whose circumference is the intersection of the surfaces of the two spheres.

**Ex. 932.** What is the area of the circle of intersection of two spheres whose radii are respectively $5^{dm}$ and $8^{dm}$, if their centers are $10^{dm}$ apart?

**Solution.** (See Fig. for Ex. 931.) Let $OB = 8^{dm}$, $PB = 5^{dm}$, and $OP = 10^{dm}$.

Denote $BD$ by $r$ and $OD$ by $x$; then, $DP = 10 - x$.

§ 350, $\overline{BD}^2 = \overline{OB}^2 - \overline{OD}^2$; that is, $r^2 = 64 - x^2$; (1)

also, $\overline{BD}^2 = \overline{PB}^2 - \overline{PD}^2$; that is, $r^2 = 25 - (10 - x)^2$;

hence, $64 - x^2 = 25 - (10 - x)^2$; whence, $x = \frac{139}{20}$,

and substituting in (1) $r^2 = 64 - (\frac{139}{20})^2 = 64 - \frac{19321}{400} = \frac{6279}{400}$;

hence, area of ⊙ of intersection $= \pi r^2 = \pi \frac{6279}{400} = 3.1416 \times \frac{6279}{400} = 49.315+ {}^{\text{sq dm}}$.

**Ex. 933.** What is the weight of an iron ball, the area of whose surface is $2^{\text{sq m}}$, the specific gravity of iron being 7.5?

**Solution.** § 726, $A = 4\pi R^2 = 2$; whence, $R^2 = \frac{1}{2\pi}$, and $R = \sqrt{\frac{1}{2\pi}}$;

then, $V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi\left(\sqrt{\frac{1}{2\pi}}\right)^3 = \frac{2}{3}\sqrt{\frac{1}{2\pi}} = .265961+ {}^{\text{cu m}} = 265961+ {}^{\text{cu cm}}$;

hence, an equal volume of water weighs $265961+ {}^{g}$, and the iron ball weighs $265961+ {}^{g} \times 7.5 = 1994707.5+ {}^{g} = 1994.7075+ {}^{Kg}$.

**Ex. 934.** If the exterior diameter of a spherical shell is 12 in., what should be the thickness of its wall in order that it may contain 696.9 cu. in.?

**Solution.** $V = \frac{1}{6}\pi D^3 = \frac{1}{6} \times 3.1416 \times 12^3 = 904.7808+$ cu. in.

Then,

$V' =$ vol. of space of interior $= 904.7808+$ cu. in. $- 696.9$ cu. in. $= 207.8808+$ cu. in.;

hence, § 745, $904.7808 : 207.8808 = 12^3 : D'^3$;

whence, $D'^3 = \frac{207.8808 \times 12^3}{904.7808} = 398.078086$, and $D' = \sqrt[3]{398.078086} = 7.35+$ in.;

hence, 12 in. $-$ 7.35+ in. $=$ 4.65$-$ in., twice the thickness of the wall; consequently, $4.65 \div 2 = 2.32+$ in. is the thickness of the wall.

**Ex. 935.** What is the weight of a hollow iron shell whose wall is 2 in. thick, if it will hold 31¼ pounds of water, the specific gravity of iron being 7.5?

**Solution.** Since 31¼ lb. is ½ the weight of a cu. ft. of water, the volume of the space within = ½ cu. ft. = 864 cu. in; but this volume $= \frac{1}{6}\pi D^3$; hence, $\frac{1}{6}\pi D^3 = 864$; whence,

$$D^3 = \frac{864 \times 6}{3.1416} = 1650.114591+, \text{ and } D = \sqrt[3]{1650.114591} = 11.8169+ \text{ in.}$$

Hence, exterior diam. $=$ 4 in. $+$ 11.8169+ in. $=$ 15.8169+ in.;

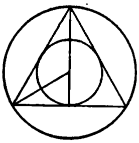
∴ entire vol. $= \frac{1}{6} \times 3.1416 \times 15.8169^3 = 2071.8759+$ cu. in.;

hence, vol. of shell $= 2071.8759 - 864 = 1207.8759+$ cu. in. $= .699+$ cu. ft.

The weight of an equal vol. of water $=$ 62.5 lb. $\times$ .699+ $=$ 43.6875+ lb.

Hence, the weight of the shell $=$ 43.6875+ lb. $\times$ 7.5 $=$ 327.656+ lb.

**Ex. 936.** An equilateral triangle revolves about its altitude. Compare the volumes of the solids generated by the triangle, the inscribed circle, and the circumscribed circle.



**Solution.** Denote the side of the $\triangle$ by $2R$, and the volumes generated by the $\triangle$, the inscribed $\odot$, and the circumscribed $\odot$ by $V$, $V'$, and $V''$, respectively.

Then, § 350, the alt. of the $\triangle = \sqrt{4R^2 - R^2} = R\sqrt{3}$,

as in Ex. 929, the radius of the inscribed $\odot = \dfrac{R}{\sqrt{3}}$,

and the radius of the circumscribed $\odot = \dfrac{2R}{\sqrt{3}}$.

Hence, § 630, $V = \frac{1}{3}\pi R^2 H = \frac{1}{3}\pi R^2 \times R\sqrt{3} = \frac{1}{3}\pi R^3\sqrt{3} = \pi\dfrac{R^3}{\sqrt{3}}$,

§ 744, $V' = \frac{4}{3}\pi\left(\dfrac{R}{\sqrt{3}}\right)^3 = \frac{4}{3}\pi\dfrac{R^3}{3\sqrt{3}} = \frac{4}{9}\pi\dfrac{R^3}{\sqrt{3}}$,

and $V'' = \frac{4}{3}\pi\left(\dfrac{2R}{\sqrt{3}}\right)^3 = \frac{4}{3}\pi\dfrac{8R^3}{3\sqrt{3}} = \frac{32}{9}\pi\dfrac{R^3}{\sqrt{3}}$.

Hence, $V : V' : V'' = \pi\dfrac{R^3}{\sqrt{3}} : \frac{4}{9}\pi\dfrac{R^3}{\sqrt{3}} : \frac{32}{9}\pi\dfrac{R^3}{\sqrt{3}} = 1 : \frac{4}{9} : \frac{32}{9} = 9 : 4 : 32.$

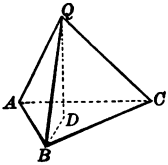
**Ex. 937.** From a sphere whose surface is 69 sq. ft. a segment of one base is cut, which has an altitude of 3 ft. What is the convex surface of the segment?

**Solution.** Surface of sph. $= 4\pi R^2 = 69$; whence, $R^2 = \dfrac{69}{4 \times 3.1416} = 5.4908+$ and $R = \sqrt{5.4908} = 2.343+$ ft.

Convex surface of segment = zone of same alt. $= 2\pi RH$

$= 2 \times 3.1416 \times 2.343 \times 3 = 44.1646+$ sq. ft.

**Ex. 938.** What is the radius of a sphere inscribed in a regular tetrahedron whose entire area is 16 sq. ft.?



**Solution.** Ex. 649, edge of face $= r\sqrt{3}$,

and apothem $= \frac{1}{2}r$;

$\therefore$ § 389, area of one face $= \frac{1}{2} \times 3r\sqrt{3} \times \frac{1}{2}r = \frac{3}{4}r^2\sqrt{3}$;

but area of one face $= \frac{1}{4}$ of 16 sq. ft. $= 4$ sq. ft.;

hence, $\frac{3}{4}r^2\sqrt{3} = 4$; whence, $r^2 = \dfrac{16}{3\sqrt{3}}$.

Then, $QD = \sqrt{\overline{QB}^2 - \overline{BD}^2} = \sqrt{(r\sqrt{3})^2 - r^2} = \sqrt{3r^2 - r^2} = \sqrt{2r^2} = \sqrt{\dfrac{32}{3\sqrt{3}}}$

Denote the radius of the inscribed sphere by $R$;

then, Ex. 831, $4R = QD = \sqrt{\dfrac{32}{3\sqrt{3}}}$, and $R = \frac{1}{4}\sqrt{\dfrac{32}{3\sqrt{3}}} = \sqrt{\dfrac{2}{3\sqrt{3}}} = .62+$ ft.

Ex. **939**. What is the area of the surface of a sphere inscribed in a regular tetrahedron whose edge is 6 in.?

**Solution.** (See Fig. for Ex. 938.) Ex. 649, side of face $= r\sqrt{3}$; but, data, side of face $= 6$ in.; hence, $r\sqrt{3} = 6$, and $r = \frac{6}{\sqrt{3}} = 2\sqrt{3}$.

Then, § 350,

the alt. $QD = \sqrt{\overline{QB}^2 - \overline{BD}^2} = \sqrt{6^2 - (2\sqrt{3})^2} = \sqrt{36 - 12} = \sqrt{24} = 2\sqrt{6}$.

Denoting the radius of the inscribed sphere by $R$,

Ex. 831, $4R = QD = 2\sqrt{6}$, and $R = \frac{1}{2}\sqrt{6}$;

then, § 726, $A = 4\pi R^2 = 4 \times 3.1416 \times (\frac{1}{2}\sqrt{6})^2 = 18.8496$ sq. in.

Ex. **940**. How much of the surface of the earth could a man see, if he were at the distance of a diameter above it?

**Solution.** § 313, $OA : OC = OC : OB$; that is, $OA : R = R : 3R$;

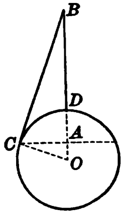
whence, $OA = \frac{R^2}{3R} = \frac{R}{3}$;

consequently, $AD = OD - OA = R - \frac{R}{3} = \frac{2R}{3}$, the altitude of the zone visible;

hence, § 728,

the area of the zone $= 2\pi RH = 2\pi R\frac{2R}{3} = \frac{4}{3}\pi R^2$;

but, § 726, the area of the surface of the earth $= 4\pi R^2$; consequently, the zone visible $= \frac{1}{3}$ of the surface of the earth.



Ex. **941**. How far from the surface of the earth must a man be in order that he may see one fifth of it?

**Solution.** (See Fig. for Ex. 940.) Area of the surface of the earth $= 4\pi R^2$; hence, the area of the zone visible $= \frac{4}{5}\pi R^2$; but area of zone $= 2\pi RH$; hence, $2\pi RH = \frac{4}{5}\pi R^2$; whence, $H = \frac{2}{5}R$, and $OA = OD - AD = R - \frac{2}{5}R = \frac{3}{5}R$.

§ 313, $OA : OC = OC : OB$; that is, $\frac{3}{5}R : R = R : OB$;

whence, $OB = \frac{5}{3}R$;

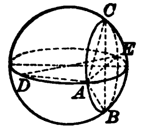
hence, $BD = OB - OD = \frac{5}{3}R - R = \frac{2}{3}R$, the distance above the earth.

Ex. **942**. All arcs of great circles drawn through the pole of a given great circle are perpendicular to its circumference.

**Proof.** Since, § 657, the planes of the arcs are perpendicular to the plane of the given great circle, § 690, the arcs are perpendicular to the circumference of the great circle.

Ex. **943**. The sum of the squares of three chords perpendicular to each other at any point in the surface of a sphere is equal to the square of the diameter.

**Data**: Three chords of a sphere, as $AB$, $AC$, and $AD$ perpendicular to each other at $A$.



**To prove** $\overline{AB}^2 + \overline{AC}^2 + \overline{AD}^2 =$ square of the diameter.

**Proof.** Pass a plane through $AB$ and $AC$.

Then, § 643, the section of the sphere made by this plane is a circle, and $\angle BAC$ is a rt. $\angle$ inscribed in it;

consequently, § 227, $BC$ is a diameter of the circle;

hence, § 349, $\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2$, or $\overline{AE}^2$, $AE$ being another diameter of the same circle, $ABC$.

Pass a plane through $AD$ and $AE$.

Since $AD \perp AB$ and $AC$, § 442, $AD \perp$ plane $ABC$, and, § 482, plane $ADE \perp$ plane $ABC$.

Since the center of the small circle is in $AE$, § 481, the axis of that circle lies in plane $ADE$; that is, plane $ADE$ passes through the center of the sphere, the section cut by it is a great circle, and, § 443, $\angle DAE$ is a rt. $\angle$ inscribed in the circle; consequently, $DE$ is a diameter of the circle and of the sphere.

Hence, $\overline{AE}^2 + \overline{AD}^2 = \overline{DE}^2$; but, $\overline{AE}^2 = \overline{AB}^2 + \overline{AC}^2$;

hence, $\overline{AB}^2 + \overline{AC}^2 + \overline{AD}^2 = \overline{DE}^2$.

Ex. **944**. If a zone of one base is a mean proportional between the remaining surface of the sphere and its total surface, how far is the base of the zone from the center of the sphere?

**Solution.** § 726, surf. of sph. $= 4\pi R^2$, and, § 728, surf. of zone $= 2\pi RH$;

consequently, the remaining surf. $= 4\pi R^2 - 2\pi RH$.

Then, data, $4\pi R^2 : 2\pi RH = 2\pi RH : 4\pi R^2 - 2\pi RH$;

$\therefore$ § 283, $2R : H = H : 2R - H$;

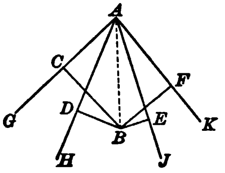
hence, § 269, $H^2 = 4R^2 - 2RH$; whence, $H = R\sqrt{5} - R$;

consequently, the required distance $= R - H = R - (R\sqrt{5} - R) = R(2 - \sqrt{5})$.

**Sch.** Since $H = R(\sqrt{5} - 1)$, which is greater than $R$, it is evident that the base of the zone does not lie *between* the pole of the zone and the center of the sphere, but *beyond* the center. This is indicated also by the negative distance, $R(2 - \sqrt{5})$.

Ex. **945**. If any number of lines in space meet in a point, the feet of the perpendiculars drawn to these lines from another point lie in the surface of a sphere.

**Data**: Any lines in space, as $AG$, $AH$, $AJ$, $AK$, etc., drawn from point $A$, and the $\perp$s drawn to them, as $BC$, $BD$, $BE$, $BF$, etc., respectively, from another point, as $B$.



**To prove** $C$, $D$, $E$, $F$, etc., lie in the surface of a sphere.

**Proof.** Draw $AB$, and circumscribe Ⓢ about the △ $ACB$, $ADB$, $AEB$, etc.

Then, since these △ are rt. △, they are inscribed in semicircles, and $AB$ is the common diameter of the Ⓢ circumscribing them ; consequently, Ax. 16, these Ⓢ are equal.

Hence, if any one of the Ⓢ revolves about $AB$ as an axis, a sphere is generated whose surface contains the points $C$, $D$, $E$, $F$, etc.

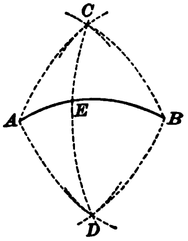
Consequently, $C$, $D$, $E$, $F$, etc., lie in the surface of a sphere.

**Ex. 946.** Bisect an arc of a great circle.

**Solution.** From the extremities $A$ and $B$ of the given arc $AB$ as poles, and with equal polar distances greater than one half arc $AB$, describe arcs intersecting as at $C$ and $D$.

§ 668, draw the great circle arc $CD$ intersecting arc $AB$ as at $E$.

Then, $AB$ is bisected at $E$.



**Proof.** Draw gt. ⊙ arcs $AC$, $BC$, $AD$, and $BD$.

In △ $ACD$ and $BCD$, $AD = BD$, $AC = BC$, and $DC$ is common ; ∴ § 712, $\angle ACD = \angle BCD$.

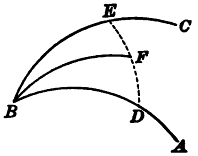
Then, in △ $AEC$ and $BEC$, $AC = BC$, $CE$ is common, and $\angle ACE = \angle BCE$ ; ∴ § 710, △ $AEC$ and $BEC$ are equal or equivalent ; that is, they are equal or symmetrical, and $AE = BE$.

**Ex. 947.** Bisect a spherical angle.

**Solution.** On the sides $BA$ and $BC$ of the given spherical angle $ABC$ measure the quadrant distances $BD$ and $BE$ respectively.

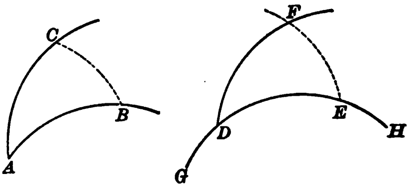
Draw the great circle arc $DE$ and, Ex. 946, bisect it as at $F$; draw the great circle arc $BF$.

Then, arc $BF$ bisects $\angle ABC$.



**Proof.** Const., △ $BDF$ and $BFE$ are mutually equilateral ; hence, § 712, △ $BDF$ and $BFE$ are mutually equiangular ; that is, $\angle DBF = \angle FBE$, or $BF$ bisects $\angle ABC$.

**Ex. 948.** At a given point in a given arc of a great circle construct a spherical angle equal to a given spherical angle.



**Solution.** With the vertex of the given $\angle A$ as a pole, describe a great circle arc cutting the sides of $\angle A$ in $B$ and $C$.

From $D$, the given point in the given arc $GH$, as a pole describe a great circle arc cutting $GH$ as at $E$, and on it take $EF = BC$.

Draw a great circle arc through $D$ and $F$.

Then, $\angle EDF$ is the $\angle$ required.

**Proof.** Const., $\triangle$ $DEF$ and $ABC$ are mutually equilateral;

hence, § 712, $\triangle$ $DEF$ and $ABC$ are mutually equiangular;

that is, $\angle EDF = \angle A$.

**Ex. 949.** Construct a spherical triangle the poles of the respective sides being given.

**Solution.** From the given poles describe great circle circumferences to intersect and form the $\triangle ABC$.

Then, $\triangle ABC$ is the $\triangle$ required.

**Proof.** The sides of $\triangle ABC$ are great circle arcs described from the given poles.

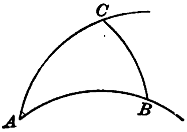
**Sch.** The circumferences of the great circles intersect to form eight spherical $\triangle$ each one of which fulfills the conditions.

**Ex. 950.** Construct a spherical triangle, having given two sides and the included angle.

**Solution.** Ex. 948, construct the spherical $\angle A =$ the given $\angle$, and on its sides lay off the arcs $AB$ and $AC =$ the given sides respectively.

Join the points $B$ and $C$ by a great circle arc.

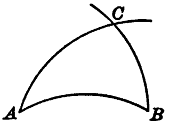
Then, $\triangle ABC$ is the $\triangle$ required.



**Ex. 951.** Construct a spherical triangle, having given a side and two adjacent angles.

**Solution.** Draw the great circle arc $AB =$ the given side; at $A$ and $B$ construct spherical $\angle$ $BAC$ and $ABC =$ respectively the given $\angle$, producing the sides to intersect as at $C$.

Then, $\triangle ABC$ is the $\triangle$ required.



**Ex. 952.** Construct a spherical triangle, having given the three sides.

**Solution.** (See Fig. for Ex. 951.) Draw the arc $AB =$ one of the given sides; with $A$ and $B$ as poles and the other given sides as arc radii respectively describe arcs to intersect as at $C$.

Join $AC$ and $BC$ by great circle arcs.

Then, $\triangle ABC$ is the $\triangle$ required.

**Ex. 953.** Construct a spherical triangle having given the three angles.

**Solution.** Produce the sides of one of the given $\angle$, and from the vertex as pole draw the arc of a great ⊙ intersecting the sides.

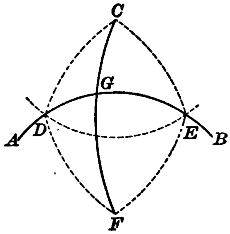
§ 689, the part of this arc included between the sides is the measure of the $\angle$, and, § 701, it is also the supplement of a side of the polar $\triangle$.

Then, subtract this arc from a semicircumference and the result is a side of the polar △.

Similarly, the other sides of the polar △ may be found and the △ constructed as in Ex. 952.

Then, draw the polar of the constructed △, and, § 700, it will be the △ required.

Ex. **954**. Draw an arc of a great circle perpendicular to a given spherical arc from a point without.



**Solution.** With the given point *C* as a pole and a polar distance greater than the distance to the given arc *AB*, describe an arc cutting *AB* as at *D* and *E*.

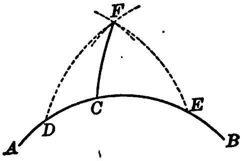
With *D* and *E* as poles and any polar distance greater than one half *DE*, describe arcs intersecting as in *F*.

Draw the great circle arc joining *C* and *F* and cutting *AB* as at *G*.

Then, arc *CF* is perpendicular to arc *AB*.

**Proof.** Proceeding as in Ex. 946, it may be shown that $\angle CGD = \angle CGE$; hence, § 681, the ∠s formed by the tangents to arcs *CF* and *AB* at *G* are equal; consequently, § 26, the tangents are perpendicular to each other; hence, arc $CF \perp$ arc $AB$.

Ex. **955**. Draw an arc of a great circle perpendicular to a given spherical arc at a point in it.



**Solution.** From the given point *C* lay off equal arcs *CD* and *CE* on the given arc *AB*.

From *D* and *E* as poles with any polar distance greater than one half *DE* describe arcs intersecting as at *F*.

Draw the great circle arc joining *C* and *F*.

Then, arc *CF* is perpendicular to arc *AB*.

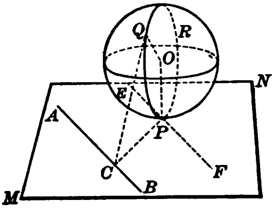
**Proof.** Draw the great circle arcs *DF* and *EF*. Const., △s *DCF* and *ECF* are mutually equilateral; hence, § 712, they are mutually equiangular; that is, $\angle FCD = \angle FCE$, and as in Ex. 954, arc $CF \perp$ arc $AB$.

Ex. **956**. Pass a plane tangent to a sphere at a given point on the surface of the sphere.

**Solution.** Draw the radius of the sphere to the given point and at that point draw two lines perpendicular to the radius; pass a plane through these lines.

Then, § 670, these lines are tangent to the sphere, and, § 674, their plane is tangent to the sphere.

**Ex. 957.** **Pass a plane tangent to a sphere through a given straight line without the sphere.**



**Solution.** Through $O$, the center of the given sphere, pass a plane $\perp AB$, the given line, intersecting $AB$ as at $C$, and the sphere in the great $\odot\ QR$.

From $C$ draw, in this plane, a tangent to $\odot\ QR$, as at $P$.

Through $AB$ and $CP$ pass a plane $MN$.

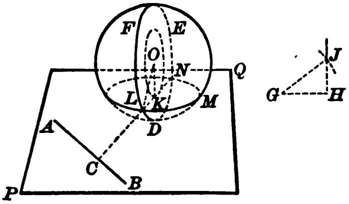
Then, $MN$ is the required plane.

**Proof.** Through $P$, in plane $MN$, draw $EF \parallel AB$, and draw $OP$.

Then, since, const., $AB \perp$ plane $CPOQ$, § 454, $EF \perp$ plane $CPOQ$; hence, § 443, $EF \perp OP$, and, § 670, $EF$ is tangent to the sphere; also $CP$ is tangent to the sphere; $\therefore$ § 674, $MN$, the plane of $CP$ and $EF$, is tangent to the sphere; that is, the plane $MN$ through $AB$ is tangent to the sphere.

**Sch.** Since there may be another tangent from $C$ to $\odot\ QR$, as $CQ$, there may be two solutions of this problem.

**Ex. 958.** **Cut a given sphere by a plane passing through a given straight line, so that the section shall have a given radius.**



**Solution.** Through $O$, the center of the given sphere, pass a plane $\perp AB$, the given line, intersecting $AB$ as at $C$, and the sphere in the great $\odot\ DEF$.

Draw $GH =$ the given radius, and at $H$ erect a $\perp$ to $GH$; from $G$ as a center, with a radius $=$ the radius of the sphere, describe an arc intersecting the $\perp$ as in $J$; then, with $HJ$ as radius and center $O$, describe a sphere intersected by the plane $CDEF$ in a $\odot$ concentric with $\odot\ DEF$.

From $C$ draw $CK$ tangent to this $\odot$, as at $K$.

Through $AB$ and $CK$ pass a plane $PQ$, cutting the given sphere in the section $LMN$, and the $\odot\ DEF$ in the chord $LN$.

Then, $PQ$ is the required plane.

**Proof.** Draw $OL$, the radius of the given sphere, and draw $OK$, the radius of the inner sphere, to the point of contact of the tangent $CK$.

Then, $\triangle LKO$ is a rt. $\triangle$, and in rt. $\triangle$s $LKO$ and $GHJ$,

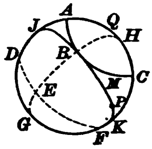
const., $OK = HJ$, and $OL = GJ$:

hence, § 123, $\triangle LKO = \triangle GHJ$, and $LK = GH =$ the given radius.

By Ex. 957, $PQ$ is tangent to the inner sphere at $K$; hence, $OK \perp$ section $LMN$, and, § 649, $OK$ is a part of the axis of $\odot\ LMN$; $\therefore$ § 652, $K$ is the center, and $LK$ is the radius of circle $LMN$.

Hence, $PQ$ is the required plane.

**Ex. 959.** Through a given point on a sphere draw a great circle tangent to a given small circle.



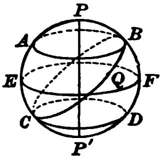
**Solution.** From $Q$, the nearer pole of the given small ⊙ $ABC$, as a pole, and with a polar distance = the polar distance of $ABC$ plus a quadrant, describe the small ⊙ $DEF$.

With the given point $P$ as a pole describe the circumference of a great ⊙ $HBEG$ cutting $ABC$ as at $B$, and $DEF$ as at $E$. With $E$ as a pole describe the circum. of gt. ⊙ $JMK$.

Then, $JMK$ is the ⊙ required.

**Proof.** Since, const., the distances $EB$ and $EP$ are quadrants, the circum. of gt. ⊙ $JMK$ described with $E$ as a pole touches $ABC$ at $B$ and passes through $P$.

**Ex. 960.** Through a given point on a sphere draw a great circle tangent to two equal small circles whose planes are parallel.



**Solution.** Ex. 959, through the given point $Q$ draw the great ⊙ $CQB$ tangent to one of the given small Ⓢ, as $AB$.

Then, $CQB$ is the ⊙ required.

**Proof.** Since, § 653, ⊙ $AB$ and ⊙ $CD$ have the same poles $P$ and $P'$, from $P$ as a pole describe the great ⊙ $EF$.

Then, § 664, arc $PF = P'F$, and, § 665, arc $PB =$ arc $P'D$;

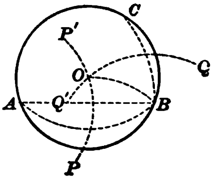
∴ arc $BF =$ arc $DF$; that is, the ⊙ $EF$ is equidistant from Ⓢ $AB$ and $CD$.

But, § 656, $CQB$ and $EF$ bisect each other; consequently, $CQB$ extends as far on one side of $EF$ as on the other, and since it touches ⊙ $AB$, it touches ⊙ $CD$.

Hence, $CQB$ is tangent to Ⓢ $AB$ and $CD$, and passes through $Q$.

**Sch.** This problem is impossible, if $Q$ is within either small ⊙.

**Ex. 961.** Describe a circle to pass through three given points on the surface of a sphere.



**Solution.** Draw a gt. ⊙ arc joining the given points $A$ and $B$, also draw a gt. ⊙ arc from $B$ to $C$ the other given point.

From $A$ and $B$ as poles with arc radius greater than $\frac{1}{2}$ arc $AB$ describe arcs intersecting as at $P$ and $P'$ and through $P$ and $P'$ pass a gt. ⊙ arc.

Similarly, draw $QQ'$ the arc of a gt. ⊙ intersecting $PP'$, as at $O$.

From $O$ as a pole with arc radius $OB$ describe the ⊙ $ABC$.

Then, $ABC$ is the ⊙ required.

**Proof.** Since, const., $P$ and $P'$ are equidistant from $A$ and $B$ measured along gt. ⊙ arcs, § 196, $P$ and $P'$ are equidistant from $A$ and $B$ measured along straight lines; Ax. 18, the center of the sphere is also equidistant from

$A$ and $B$; consequently, §§ 430, 452, the plane of arc $PP' \perp$ chord $AB$ at its middle point; hence, § 452, $O$ is equidistant from $A$ and $B$ along straight lines, and consequently, § 196, along gt. ⊙ arcs.

Similarly, $O$ is equidistant from $B$ and $C$.

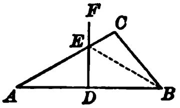
Hence, the circumference of a ⊙ described from $O$ as a pole with $OB$ as arc radius, passes through $A$, $B$, and $C$.

**Ex. 962. Circumscribe a circle about a given spherical triangle.**

**Solution. Pass a ⊙ through the three vertices of the △ as in. Ex. 961.**

## EXERCISES FOR REVIEW

**Ex. 1. The perpendicular erected at the middle point of one side of a triangle meets the longer of the other two sides.**



**Data:** $\triangle ABC$, and $FD \perp AB$ at its middle point, meeting a side, as $AC$, at $E$.

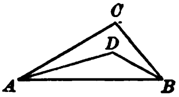
**To prove** $AC > BC$.

**Proof.** Draw $BE$.

Then, § 125, $BE + EC > BC$; but, § 103, $BE = AE$;

∴ substituting, $AE + EC > BC$; that is, $AE > BC$.

Ex. 2. Of the bisectors of two unequal angles of a triangle, produced to the point of intersection, the bisector of the smaller angle is the longer.



**Data:** $\triangle ABC$ in which $\angle A$ is less than $\angle B$, also the bisectors of these $\angle$s meeting in $D$.

**To prove** $AD > BD$.

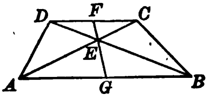
**Proof.** Since, data, $\angle A$ is less than $\angle B$, $\angle DAB$, the half of $\angle A$, is less than $\angle ABD$, the half of $\angle B$; hence, § 127, $AD > BD$.

Ex. 3. The straight lines which join the middle points of the opposite sides of any quadrilateral bisect each other.

**Proof.** Join the middle points in succession.

Then, Ex. 181, a ▱ is formed whose diagonals are the given lines; hence, § 154, the given lines bisect each other.

Ex. 4. If a line is drawn from the middle point of one base of a trapezoid to pass through the intersection of the diagonals, it will bisect the other base.



**Data:** Trapezoid $ABCD$; $E$ the intersection of its diagonals and the line from $F$, the middle point of $DC$, drawn through $E$ and meeting $AB$ in $G$.

**To prove** $AB$ bisected at $G$.

**Proof.** § 73, $\angle EDF = \angle EBG$, and, § 59, $\angle DEF = \angle BEG$;

∴ § 301, $\triangle$s $DEF$ and $BEG$ are similar, and $DF : BG = EF : EG$;

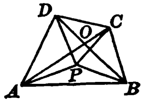
similarly, from $\triangle$ $EFC$ and $AEG$, $FC : AG = EF : EG$;

$\therefore$ $DF : BG = FC : AG$; but, data, $DF = FC$; $\therefore$ § 272, $BG = AG$;
that is, $AB$ is bisected at $G$.

Ex. 5. If the opposite sides of a pentagon are produced to intersect, the sum of the angles at the vertices of the triangles thus formed is equal to two right angles.

**Proof.** By § 167, the sum of the base $\angle$ of these $\triangle = 4$ rt. $\angle \times 2 = 8$ rt. $\angle$; and since there are five $\triangle$, § 110, the sum of all their $\angle = 2$ rt. $\angle \times 5 = 10$ rt. $\angle$; hence, the sum of the $\angle$ at the vertices $= 10$ rt. $\angle - 8$ rt. $\angle = 2$ rt. $\angle$.

Ex. 6. The sum of the four lines drawn from the vertices of any quadrilateral to any point, except the intersection of the diagonals, is greater than the sum of the diagonals.

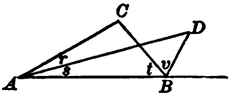


**Data:** Quad. $ABCD$ whose diagonals $AC$ and $BD$ intersect at $O$, and the lines $AP$, $BP$, $CP$, and $DP$ drawn from the vertices to any point (except $O$) as $P$.

**To prove** $AP + BP + CP + DP > AC + BD$.

**Proof.** § 125, $AP + CP > AC$, and $DP + BP > BD$;
hence, $AP + BP + CP + DP > AC + BD$.

Ex. 7. If the internal bisector of one base angle of a triangle and the external bisector of the other base angle are produced until they meet, the angle included between them is equal to half the vertical angle of the triangle.

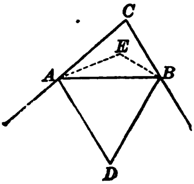


**Data:** $\triangle ABC$, and the bisectors $AD$ and $BD$ (meeting at $D$) of the int. $\angle A$ and the ext. $\angle$ at $B$, respectively.

**To prove** $\angle D = \frac{1}{2} \angle C$.

**Proof.** § 110, $\angle D = 2$ rt. $\angle - (\angle s + \angle t + \angle v)$;
but, data, $\angle s = \angle r$, and $\angle v = \frac{1}{2}(2 \text{ rt. } \angle - \angle t)$;
hence, $\angle D = 2 \text{ rt. } \angle - (\angle r + \angle t + 1 \text{ rt. } \angle - \frac{1}{2}\angle t) = 1 \text{ rt. } \angle - \angle r - \frac{1}{2}\angle t$.
Also, $\angle C = 2 \text{ rt. } \angle - (\angle r + \angle s + \angle t) = 2(1 \text{ rt. } \angle - \angle r - \frac{1}{2}\angle t)$.
Hence, $\angle D = \frac{1}{2}\angle C$.

Ex. 8. The angle contained by the bisectors of two exterior angles of any triangle is equal to half the sum of the two adjacent interior angles.



**Data:** $\triangle ABC$, and the bisectors $AD$ and $BD$ of two of its ext. $\angle$, $A$ and $B$.

**To prove** $\angle ADB = \frac{1}{2}(\angle CAB + \angle ABC)$.

**Proof.** Draw $AE$ and $BE$ to bisect $\angle$ $CAB$ and $ABC$.

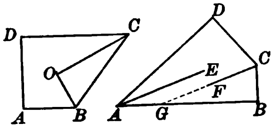
Then, Ex. 14, $\angle$s $DAE$ and $DBE$ are rt. $\angle$s; $\therefore$ $\angle D$ is the sup. of $\angle E$;
but, § 110, $\angle EAB + \angle ABE$ is the sup. of $\angle E$;
hence, § 54, $\angle D = \angle EAB + \angle ABE$;
but, const., $\angle EAB = \frac{1}{2}\angle CAB$, and $\angle ABE = \frac{1}{2}\angle ABC$;
hence, $\angle D = \frac{1}{2}(\angle CAB + \angle ABC)$.

**Ex. 9.** If each of two angles of a quadrilateral is a right angle, the bisectors of the other angles are either perpendicular or parallel to each other.

**Case I. Data:** Quad. $ABCD$ in which *consecutive* $\angle$s $A$ and $D$ are rt. $\angle$s, and the bisectors $BO$ and $CO$ of its other $\angle$s meeting in $O$.

**To prove** $BO \perp CO$.

**Proof.** § 166, the sum of the $\angle$s of $ABCD = 4$ rt. $\angle$s, and since, data, $\angle A + \angle D = 2$ rt. $\angle$s, Ax. 3, $\angle ABC + \angle BCD = 2$ rt. $\angle$s; hence, in $\triangle BCO$, data, Ax. 7, $\angle OBC + \angle BCO = 1$ rt. $\angle$,
consequently, § 110, $\angle BOC$ is a rt. $\angle$;
that is, $BO \perp CO$.



Case I Case II

**Case II. Data:** Quad. $ABCD$ in which *opposite* $\angle$s $B$ and $D$ are rt. $\angle$s, and $AE$ and $CF$ are the bisectors of the other $\angle$s.

**To prove** $AE$ and $CF$ parallel.

**Proof.** Produce $CF$ to meet $AB$ as at $G$.

As in Case I it may be shown that $\angle BAE + \angle BCF = 1$ rt. $\angle$; that is, § 31, $\angle BAE$ is the comp. of $\angle BCG$; but, § 111, $\angle BGC$ is the comp. of $\angle BCG$; hence, § 54, $\angle BAE = \angle BGC$; consequently, § 77, $AE \parallel GC$; that is, $AE$ and $CF$ are parallel.

**Ex. 10.** If the side $CB$ of the triangle $ABC$ is greater than the side $CA$, and $CA$ is produced to $D$ and $CB$ to $E$, making $AD$ and $BE$ equal, $AE$ is greater than $DB$.

**Proof.** Since $CB > CA$, § 126, $\angle CAB > \angle ABC$;
but $\angle CAB + \angle DAB = 2$ rt. $\angle$s $= \angle ABC + \angle ABE$;
hence, $\angle DAB < \angle ABE$.

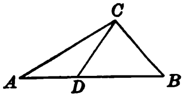
Then, in $\triangle$s $AEB$ and $DBA$, $AD = BE$, $AB$ is common,
and $\angle ABE > \angle DAB$; hence, § 129, $AE > DB$.

**Ex. 11.** In the $\triangle ABC$ a straight line $AD$ is drawn perpendicular to the straight line $BD$, which bisects angle $B$. Prove that a straight line through $D$, parallel to $BC$, bisects $AC$.

**Proof.** Let the line through $D$, parallel to $BC$, cut $AB$ at $E$ and $AC$ at $F$.
§ 73, $\angle EDB = \angle DBC$, and, data, $\angle EBD = \angle DBC$;
$\therefore$ $\angle EDB = \angle EBD$, and, § 118, $BE = DE$.

Data, § 31, $\angle ADE$ is the comp. of $\angle EDB$,
and, § 111, $\angle DAE$ is the comp. of $\angle EBD$;
and since $\angle EDB = \angle EBD$, § 54, $\angle ADE = \angle DAE$, and $AE = DE$;
hence, Ax. 1, $AE = BE$; consequently, § 158, $EF$ bisects $AC$.

Ex. **12**. If one side of a triangle is greater than the other, any line from the vertex of the included angle to the base is less than the longer side.



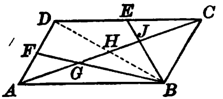
**Data**: $\triangle ABC$ in which $AC > BC$; also $CD$ any line from $C$ to $AB$.

**To prove** $CD < AC$.

**Proof.** Since, in $\triangle ABC$, $AC > BC$, § 126, $\angle B > \angle A$.

Since, § 115, $\angle ADC = \angle B + \angle BCD$, $\angle ADC > B$;
consequently, $\angle ADC > \angle A$;
hence, in $\triangle ADC$, § 127, $AC > CD$, or $CD < AC$.

Ex. **13**. Lines drawn from one vertex of a parallelogram to the middle points of the opposite sides trisect a diagonal.



**Data:** $\square ABCD$ and the lines $BE$ and $BF$ from $B$ to the middle points of $DC$ and $AD$ respectively cutting diagonal $AC$ in $J$ and $G$.

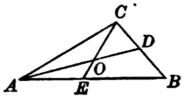
**To prove** $BE$ and $BF$ trisect $AC$.

**Proof.** Draw diagonal $BD$ cutting $AC$ as at $H$.

Then, since, § 154, $AC$ and $BD$ bisect each other, $CH$ is a median of $\triangle DBC$, and, data, $BE$ is also a median;
hence, § 168, $CJ = \frac{2}{3}CH$, and, similarly, $AG = \frac{2}{3}AH$;
but $AH = CH$; $\therefore$ $AG = CJ$, and $AG + CJ = 2\,AG = \frac{2}{3}(AH + CH) = \frac{2}{3}AC$;
hence, $AG = \frac{1}{3}AC$, also $CJ = \frac{1}{3}AC$; consequently, $GJ = \frac{1}{3}AC$.

Hence, $BE$ and $BF$ trisect $AC$.

Ex. **14**. No two straight lines drawn from two vertices of a triangle and terminated by the opposite sides can bisect each other.



**Data**: $\triangle ABC$, and any two lines as $AD$ and $CE$ from two vertices to the opposite sides intersecting each other in $O$.

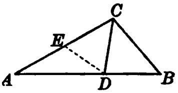
**To prove** $AD$ and $CE$ do *not* bisect each other.

**Proof.** Suppose that $AD$ and $CE$ *do* bisect each other.

Then, hyp., $AO = DO$, $EO = CO$, and, § 59, $\angle AOE = \angle COD$;
$\therefore$ § 100, $\triangle AEO = \triangle CDO$, and, § 108, $\angle AEO = \angle OCD$;
hence, § 75, $AB \parallel BC$, which is absurd, for $AB$ and $BC$ are sides of a $\triangle$.

Hence, the hyp. that $AD$ and $CE$ bisect each other is untenable; consequently, no two lines drawn from two vertices to the opposite sides bisect each other.

**Ex. 15.** The base of a triangle whose sides are unequal is divided into two parts by a straight line bisecting the vertical angle. Prove that the greater part is adjacent to the greater side.



**Data**: $\triangle ABC$ having $AC > BC$ and $CD$ the bisector of $\angle C$ meeting $AB$ in $D$.

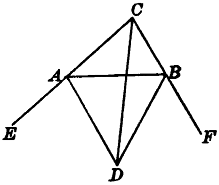
**To prove** $AD > BD$.

**Proof.** Take $CE = CB$ and draw $ED$.

In $\triangle$s $EDC$ and $DBC$, const., $CE = CB$, $CD$ is common,
and, data, $\angle DCE = \angle DCB$; $\therefore$ § 100, $\triangle EDC = \triangle DBC$.

Since, § 115, $\angle AED = \angle EDC + \angle ECD$, $\angle AED > \angle EDC$;
but, § 108, $\angle EDC = \angle BDC$; hence, $\angle AED > \angle BDC$;
as above, $\angle BDC > \angle CAD$; $\therefore$ $\angle AED > \angle CAD$, or $\angle EAD$;
hence, § 127, $AD > ED$; but, § 108, $ED = BD$; $\therefore$ $AD > BD$.

**Ex. 16.** If two exterior angles of a triangle are bisected, and from the point of intersection of the bisectors a straight line is drawn to the vertex of the third angle, this line bisects that angle.



**Data**: $\triangle ABC$; the bisectors $AD$ and $BD$ of two of its exterior $\angle$s, as at $A$ and $B$, meeting in $D$; and line $CD$.

**To prove** $CD$ bisects $\angle ACB$.

**Proof.** Let $CE$ and $CF$ be the sides $CA$ and $CB$ produced.

§ 134, $D$ is equidistant from $AE$ and $AB$, and also from $AB$ and $BF$;
hence, $D$ is equidistant from $CE$ and $CF$;
consequently, § 135, $D$ lies in the bisector of $\angle ECF$;
that is, $CD$ bisects $\angle ACB$.

**Ex. 17.** $ABC$ is a triangle; $D$ is the middle point of $BC$, and $E$ of $AD$; $BE$ produced meets $AC$ in $F$. Prove that $AC$ is trisected in $F$.

**Proof.** Draw $DG \parallel BF$, meeting $AC$ as in $G$.

Then, § 158, $GC = FG$, and since, data, $E$ is the middle point of $AD$, § 158, $AF = FG$; hence, $AF = FG = GC$; that is, $AC$ is trisected in $F$.

**Ex. 18.** In the triangle $ABC$ the sides $AB$, $BC$, and $CA$ are trisected at the consecutive points $D$ and $E$, $F$ and $G$, and $H$ and $K$ respectively. Prove that the lines $EF$, $GH$, and $KD$, when produced, form a triangle equal to $ABC$.

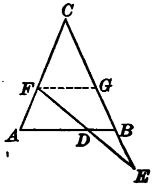
**Proof.** Let lines $EF$, $GH$, and $KD$ when produced intersect to form $\triangle LMN$.

Since, § 291, the sides of $\triangle LMN$ are parallel to sides of $\triangle ABC$ respectively, § 81, the $\angle$s of $\triangle LMN$ are equal respectively to the angles of $\triangle ABC$.

**Draw** $HE$ and $DG$.

§ 291, $HE \parallel CB$, and $ND \parallel CB$; $\therefore$ § 80, $ND \parallel HE$,
and since $NH \parallel DE$, § 140, $DEHN$ is a parallelogram;
$\therefore NH = DE = \frac{1}{3}AB$; similarly, $HG = EB = \frac{1}{3}AB$, and $GM = DE = \frac{1}{3}AB$;
hence, $NH + HG + GM = \frac{1}{3}AB + \frac{1}{3}AB + \frac{1}{3}AB$; that is, $NM = AB$,
and since $\angle N = \angle B$, and $\angle M = \angle A$, § 102, $\triangle LMN = \triangle ABC$.

Ex. **19**. If one of the equal sides of an isosceles triangle is produced below the base to a certain length, if an equal length is cut off above the base from the other equal side, and if the two points are joined by a straight line, this line is bisected by the base.



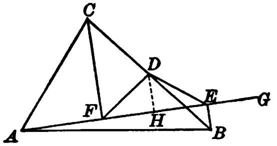
**Data**: Isos. $\triangle ABC$; one of the equal sides $CB$ produced to $E$; $AF = BE$ cut off on the other equal side $AC$; and the line $EF$ intersecting $AB$ in $D$.

**To prove** $EF$ bisected by $AB$.

**Proof.** Draw $FG \parallel AB$, meeting $BC$ as at $G$.

Then, Ex. 125, $\angle CFG = \angle CGF$;
$\therefore$ § 118, $GC = FC$; but, data, $BC = AC$;
hence, Ax. 3, $BG = AF$; but, data, $BE = AF$; $\therefore BG = BE$;
hence, in $\triangle FGE$, § 158, $FD = DE$; that is, $EF$ is bisected by $AB$.

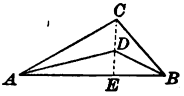
Ex. **20**. $ABC$ is a triangle, and $BE$ and $CF$ are drawn perpendicular to $AG$, a line through $A$; $D$ is the middle point of $BC$. Show that $FD$ equals $ED$.



**Proof.** Draw $DH \parallel CF$, meeting $AG$ as at $H$.

§ 71, $BE \parallel CF$; $\therefore BE \parallel HD$,
and since, data, $CD = DB$, § 157, $FH = HE$;
§ 72, $DH \perp EF$; hence, § 103, $FD = ED$.

Ex. **21**. The angle contained by the bisectors of the base angles of any triangle is equal to the vertical angle of the triangle plus half the sum of the base angles.



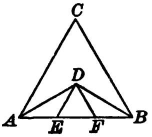
**Data**: $\triangle ABC$, and the bisectors $AD$ and $BD$ of its base $\angle$s, produced to meet as at $D$.

**To prove** $\angle ADB = \angle ACB + \frac{1}{2}(\angle BAC + \angle ABC)$.

**Proof.** Draw $CD$ and produce it to meet $AB$ as at $E$.

Then, § 115, $\angle BDE = \angle BCD + \angle DBC$,
and $\angle ADE = \angle ACD + \angle DAC$;
hence, Ax. 2, $\angle ADE + \angle BDE = \angle ACD + \angle BCD + \angle DAC + \angle DBC$; (1)
but $\angle ADE + \angle BDE = \angle ADB$, $\angle ACD + \angle BCD = \angle ACB$,
and $\angle DAC + \angle DBC = \frac{1}{2}(BAC + \angle ABC)$;
$\therefore$ substituting in (1) $\angle ADB = \angle ACB + \frac{1}{2}(BAC + \angle ABC)$.

**Ex. 22.** The bisectors of two angles of an equilateral triangle intersect, and from their point of intersection lines are drawn parallel to any two sides. Prove that these lines trisect the third side.



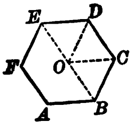
**Data:** Equilateral $\triangle ABC$; $D$ the intersection of the bisectors, $AD$ and $BD$, of two of its $\angle$s; also lines from $D$, as $DE$ and $DF$, parallel to two of its sides, $CA$ and $CB$ respectively, meeting $AB$ in $E$ and $F$.

**To prove** $DE$ and $DF$ trisect $AB$.

**Proof.** § 117, $\triangle ABC$ is equiangular, and since, § 81, the $\angle$s of $\triangle EFD$ are equal respectively to those of $\triangle ABC$, $\triangle EFD$ is equiangular; $\therefore$ § 119, $\triangle EFD$ is equilateral.

§ 73, $\angle FDB = \angle DBC$, and, data, $\angle FBD = \angle DBC$;
hence, $\angle FDB = \angle FBD$; $\therefore$ § 118, $FB = FD = EF$;
similarly, $AE = EF$; hence, $AE = EF = FB = \frac{1}{3} AB$;
that is, $DE$ and $DF$ trisect $AB$.

**Ex. 23.** The opposite sides of a regular hexagon are parallel.



**Data:** Regular hexagon $ABCDEF$, and any two opposite sides, as $AB$ and $ED$.

**To prove** $AB \parallel ED$.

**Proof.** Draw the radii $OB$, $OC$, $OD$, and $OE$.

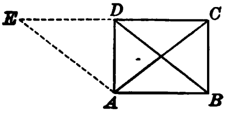
Then, § 383, each of the $\angle$s $BOC$, $COD$, $DOE$ is equal to

$$4 \text{ rt. } \angle\text{s} \div 6 = \tfrac{2}{3} \text{ rt. } \angle; \therefore \angle BOC + \angle COD + \angle DOE = 2 \text{ rt. } \angle\text{s},$$

or Ax. 9, $\angle BOC + \angle COE = 2$ rt. $\angle$s; hence, § 58, $BOE$ is a straight line.

§ 374, $\angle ABC = \angle DEF$; $\therefore$ § 382, Ax. 7, $\angle ABE = \angle DEB$;
hence, § 75, $AB \parallel ED$.

**Ex. 24.** If in a quadrilateral the diagonals are equal and two sides are parallel, the other sides are equal.



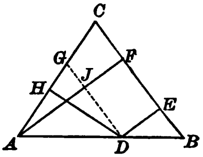
**Data:** Quadrilateral $ABCD$, whose diagonals $AC$ and $BD$ are equal, and two sides $AB$ and $DC$ parallel.

**To prove** $AD = BC$.

**Proof.** Draw $AE \parallel BD$, meeting $CD$ produced as in $E$.

Then, § 140, $ABDE$ is a parallelogram,
and, § 153, $AE = BD$; but, data, $AC = BD$;
hence, $AE = AC$; $\therefore$ § 116, $\angle AEC = \angle ACE$;
but, § 73, $\angle CAB = \angle ACE$; $\therefore$ Ax. 1, $\angle AEC = \angle CAB$,
and since $AE = AC$, and, § 153, $ED = AB$,
§ 100, $\triangle ADE = \triangle ABC$, and $AD = BC$.

**Ex. 25.** If from any point in the base of an isosceles triangle perpendiculars are drawn to the equal sides, their sum is equal to the perpendicular drawn from either extremity of the base to the opposite side.



**Data:** Isos. $\triangle ABC$; $D$ any point of base $AB$;

$DE \perp BC$; $DH \perp AC$; and $AF \perp BC$.

**To prove** $DE + DH = AF$.

**Proof.** Draw $DG \parallel BC$, meeting $AC$ as in $G$, and cutting $AF$ as in $J$.

Then, § 76, $\angle ADG = \angle ABC$,

and, § 116, $\angle GAD = \angle ABC$;

hence, $\angle GAD = \angle ADG$, and, § 118, $DG = AG$.

Then, in rt. $\triangle$s $DHG$ and $AJG$,

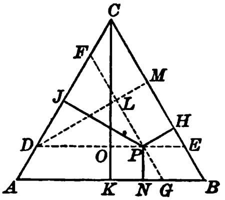
$DG = AG$, and $\angle HGD$ is common;

$\therefore$ § 114, $\triangle DHG = \triangle AJG$, and $DH = AJ$.

§ 71, $DE \parallel AF$, and, const., $DG \parallel BC$; $\therefore$ § 151, $DE = JF$.

Hence, $DH + DE = AJ + JF = AF$.

**Ex. 26.** The sum of the perpendiculars from any point within an equilateral triangle to its sides is equal to the altitude.



**Data:** Equilateral $\triangle ABC$; any point within, as $P$; and the $\perp$s $PN$, $PH$, and $PJ$ from $P$ to the sides $AB$, $BC$, and $AC$, respectively; also the altitude $CK$.

**To prove** $PN + PH + PJ = CK$.

**Proof.** Through $P$ draw $DE \parallel AB$, cutting $CK$ at $O$; $GF \parallel BC$; and $DM \perp BC$, cutting $GF$ at $L$.

Since, in $\triangle$s $DEC$ and $ABC$,

§ 76, $\angle CDE = \angle A$, $\angle CED = \angle B$, and $\angle C$ is common,

and since, § 117, $\triangle ABC$ is equiangular, $\triangle DEC$ is equiangular; hence, § 119, $\triangle DEC$ is equilateral; similarly, $\triangle DPF$ is equilateral.

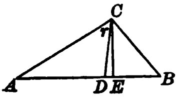
§ 71, $PN \parallel OK$; $\therefore$ § 151, $PN = OK$; similarly, $PH = LM$;

§ 94, $PJ = DL$, and $DM = CO$.

Then, Ax. 2,

$$PN + PH + PJ = OK + LM + DL = OK + DM = OK + CO = CK.$$

**Ex. 27.** If from the vertex of any triangle two lines are drawn, one of which bisects the angle at the vertex, and the other is perpendicular to the base, the angle between these lines is half the difference of the angles at the base of the triangle.



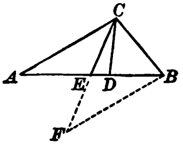
**Data:** $\triangle ABC$, in which $\angle B > \angle A$; $CD$, the bisector of vertical $\angle C$; and $CE \perp$ the base $AB$.

**To prove** $\angle DCE = \frac{1}{2}(\angle B - \angle A)$.

**Proof.** §§ 111, 115, $\angle DCE = 1 \text{ rt.}\angle - \angle EDC = 1 \text{ rt.}\angle - (\angle A + \angle r)$; (1)
but $\angle r = \frac{1}{2}\angle ACB = \frac{1}{2}[2 \text{ rt.} \angle s - (\angle A + \angle B)] = 1 \text{ rt.}\angle - \frac{1}{2}(\angle A + \angle B)$;
∴ substituting,

$$\angle DCE = 1 \text{ rt.}\angle - [\angle A + 1 \text{ rt.}\angle - \tfrac{1}{2}(\angle A + \angle B)] = \tfrac{1}{2}(\angle B - \angle A).$$

**Ex. 28.** In any triangle, the sides of the vertical angle being unequal, the median drawn from the vertical angle lies between the bisector of that angle and the longer side.



**Data:** $\triangle ABC$, in which $AC > BC$; $CD$, the bisector of the vertical $\angle$; and $CE$, the median from the vertex.

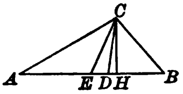
**To prove** that $CE$ lies between $AC$ and $CD$.

**Proof.** Produce $CE$ to $F$, making $EF = CE$, and draw $FB$.

Data, $AE = BE$, const., $CE = EF$, and, § 59, $\angle AEC = \angle FEB$;
∴ § 100, $\triangle ACE = \triangle EFB$, $AC = FB$, and $\angle ACE = \angle BFC$.

Data, $AC > BC$; ∴ $FB > BC$; hence, § 126, $\angle BCF > \angle BFC$;
that is, $\angle BCE > \angle ACE$; consequently, $\angle BCE > \frac{1}{2}\angle ACB$, or $\angle BCD$;
hence, $CE$ lies between $AC$ and $CD$.

**Ex. 29.** In any triangle, the sides of the vertical angle being unequal, the bisector of that angle lies between the median and the perpendicular drawn from the vertex to the base.

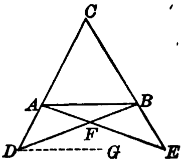


**Data:** $\triangle ABC$, in which $AC > BC$; $CD$, the bisector of the vertical $\angle$; $CE$, the median from the vertex; and $CH$, the perpendicular from the vertex.

**To prove** $CD$ lies between $CE$ and $CH$.

**Proof.** § 111, $\angle BCH$ is comp. of $\angle B$, and $\angle ACH$ is comp. of $\angle A$;
but, § 126, $\angle B > \angle A$; ∴ $\angle BCH < \angle ACH$;
hence, $\angle BCH < \frac{1}{2}\angle ACB$, or $\angle BCD$;
consequently, $CH$ lies between $CD$ and $BC$.
Ex. 28 Rev., $CE$ lies between $CD$ and $AC$.
Hence, $CD$ lies between $CE$ and $CH$.

**Ex. 30.** Lines are drawn through the extremities of the base of an isosceles triangle, making angles with it, on the side opposite the vertex, each equal to one third of a base angle of the triangle, and meeting the sides produced. Prove that the three triangles thus formed are isosceles.



**Data:** Isos. $\triangle ABC$; lines $AE$ and $BD$, forming $\angle s$ with $AB$ each equal to $\frac{1}{3}\angle ABC$, intersecting each other in $F$ and the sides produced in $E$ and $D$ respectively.

**To prove** $\triangle s$ $AFB$, $FEB$, and $DFA$ isosceles.

**Proof.** Data, Ax. 1, $\angle FAB = \angle FBA$; hence, § 118, $\triangle AFB$ is isosceles.

Draw $DG \parallel AB$.

Then, § 76, $\angle CDG = \angle CAB$, and, § 73, $\angle BDG = \angle ABD$;

$\therefore$ Ax. 3, $\angle CDG - \angle BDG = \angle CAB - \angle ABD$;

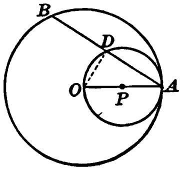
that is, $\angle ADF = \frac{2}{3}\angle CAB$;

but, § 115, data, $\angle DFA = \angle FAB + \angle FBA = \frac{2}{3}\angle CAB$;

hence, Ax. 1, $\angle ADF = \angle DFA$, and, § 118, $\triangle DFA$ is isosceles.

Similarly it may be shown that $\triangle FEB$ is isosceles.

Ex. **31**. If two circumferences are tangent internally and the radius of the larger is the diameter of the smaller, any chord of the larger drawn from the point of contact is bisected by the circumference of the smaller.



**Data:** Circumferences whose centers are $O$ and $P$ tangent internally at $A$, the radius of the larger being equal to the diameter of the smaller; also any chord of the larger, drawn from $A$, as $AB$, cut by the circumference of the smaller as at $D$.

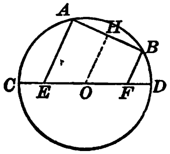
**To prove** $AB$ bisected at $D$.

**Proof.** Draw $OD$.

Then, § 227, $\angle ODA$ is a rt. $\angle$; that is, $OD \perp AB$;

hence, § 200, $AB$ is bisected at $D$.

Ex. **32**. If perpendiculars are drawn to any chord at its extremities and produced to intersect a diameter of the circle, the points of intersection are equally distant from the center.



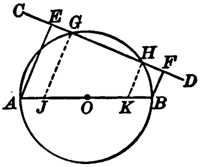
**Data:** Chord $AB$ of the ⊙ whose center is $O$; also $AE$ and $BF \perp AB$, meeting diam. $CD$ in $E$ and $F$ respectively.

**To prove** $EO = FO$.

**Proof.** Draw $OH \perp AB$.

Then, § 71, $OH \parallel EA \parallel FB$, and since, § 200, $AH = BH$, § 157, $EO = FO$.

Ex. **33**. If perpendiculars are drawn from the ends of a diameter of a circle upon any secant, their feet are equally distant from the points in which the secant intersects the circumference.



**Data:** Diam. $AB$ of the ⊙ whose center is $O$; the secant $CD$ cutting the circumference at $G$ and $H$; also $AE$ and $BF \perp CD$.

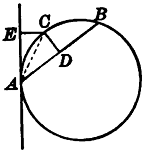
**To prove** $EG = HF$.

**Proof.** Draw $GJ$ and $HK \perp CD$, meeting $AB$ as in $J$ and $K$ respectively.

Then, Ex. 32 Rev., $OJ = OK$; but, Ax. 14, $OA = OB$;

$\therefore$ Ax. 3, $AJ = KB$, and, since, § 71, $AE \parallel JG \parallel KH \parallel BF$, $EG = HF$.

**Ex. 34.** Given an arc of a circumference, the chord subtended by it, and the tangent at one extremity. Prove that the perpendiculars dropped from the middle point of the arc upon the tangent and chord, respectively, are equal.



**Data:** Arc $AB$ and its chord; tangent $AE$; and the $\perp$s $CE$ and $CD$ from $C$, the middle point of arc $AB$, to $AE$ and chord $AB$.

**To prove** $CE = CD$.

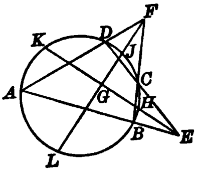
**Proof.** Draw $AC$.

Then, in rt. $\triangle$s $AEC$ and $ADC$, $AC$ is common,

and, §§ 231, 225, $\angle EAC = \angle DAC$ (being measured by one half of equal arcs $AC$ and $BC$ respectively);

hence, § 114, $\triangle AEC = \triangle ADC$, and $CE = CD$.

**Ex. 35.** The bisectors of the angles contained by the opposite sides (produced) of an inscribed quadrilateral intersect at right angles.



**Data:** Opposite sides of inscribed quadrilateral $ABCD$ produced to intersect at $E$ and $F$, and the bisectors, $EK$ and $FL$, of $\angle$s $E$ and $F$ respectively intersecting at $G$.

**To prove** $\angle EGF$ a rt. $\angle$.

**Proof.** Since $\angle AFL = \angle BFL$,

§ 234, $\frac{1}{2}(\text{arc } AL - \text{arc } DJ) = \frac{1}{2}(\text{arc } LB - \text{arc } JC)$;

hence, $\text{arc } AL - \text{arc } DJ = \text{arc } LB - \text{arc } JC$;

similarly, $\text{arc } AK - \text{arc } BH = \text{arc } KD - \text{arc } HC$;

adding, $\text{arc } LK - \text{arc } DJ - \text{arc } BH = \text{arc } LB + \text{arc } KD - \text{arc } HJ$;

transposing, $\text{arc } HJ + \text{arc } LK = \text{arc } LB + \text{arc } BH + \text{arc } KD + \text{arc } DJ$;

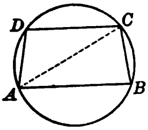
that is, $\text{arc } HJ + \text{arc } LK = \text{arc } LH + \text{arc } KJ$.

But, § 231, $\angle EGF$ is measured by $\frac{1}{2}(\text{arc } HJ + \text{arc } LK)$,

and $\angle LGE$ is measured by $\frac{1}{2}(\text{arc } LH + \text{arc } KJ)$;

consequently, $\angle EGF = \angle LGE$; hence, § 26, $\angle EGF$ is a rt. $\angle$.

**Ex. 36.** If two opposite sides of an inscribed quadrilateral are equal, the other two sides are parallel.



**Data:** Inscribed quadrilateral $ABCD$, whose opposite sides $AD$ and $BC$ are equal.

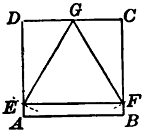
**To prove** $AB \parallel DC$.

**Proof.** Draw $AC$.

Since $AD = BC$, § 196, arc $AD =$ arc $BC$;

hence, § 225, $\angle ACD = \angle BAC$; $\therefore$ § 75, $AB \parallel DC$.

Ex. **37**. In a given square, inscribe an equilateral triangle having its vertex in the middle of a side of the square.



**Solution.** Find the middle point $G$ of the side $DC$ of the given square $ABCD$.

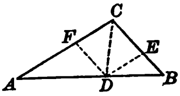
With $G$ as a center and a radius = a side $AB$, describe arcs cutting sides $AD$ and $BC$ as in $E$ and $F$ respectively.

Draw $EF$, $FG$, and $EG$.

Then, $\triangle EFG$ is the $\triangle$ required.

**Proof.** Const., $EG = FG$, $DG = CG$; $\therefore$ § 123, $\triangle EGD = \triangle FGC$, and $ED = FC$; hence, Ax. 3, $AE = BF$, and since $AE \parallel BF$, § 150, $ABFE$ is a ▱, and, § 153, $EF = AB$; but $EG = FG = AB$; that is, $\triangle EFG$ is an equilateral $\triangle$.

Ex. **38**. Find, in a side of a triangle, a point from which straight lines, drawn parallel to the other sides of the triangle and terminated by them, are equal.



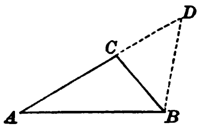
**Solution.** Bisect the $\angle C$ opposite the given side $AB$ of the given $\triangle ABC$, and produce the bisector to meet $AB$ as in $D$.

Then, $D$ is the required point.

**Proof.** Draw $DE \parallel AC$ and $DF \parallel BC$ meeting $BC$ and $AC$ as in $E$ and $F$ respectively.

Then, § 140, $DECF$ is a ▱, and $DE = FC$;
but, § 73, $\angle FDC = \angle DCE$, and, const., $\angle FCD = \angle DCE$;
$\therefore$ $\angle FDC = \angle FCD$, and, § 118, $DF = FC$; hence, Ax. 1, $DE = DF$.

Ex. **39**. Construct a triangle, having given the base, one of the angles at the base, and the sum of the other two sides.



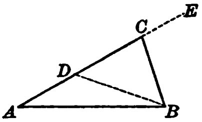
**Solution.** Draw $AB$ = the given base, and at $A$ construct $\angle BAD$ = the given $\angle$ making $AD$ = the given sum of sides.

Draw $BD$, and construct $\angle DBC = \angle D$, the side $BC$ meeting $AD$ as in $C$.

Then, $\triangle ABC$ is the $\triangle$ required.

**Proof.** Since, const., $\angle DBC = \angle D$, § 118, $BC = CD$;
hence, $AC + BC = AC + CD = AD$ = the given sum of sides.

Ex. **40**. Construct a triangle, having given the base, one of the angles at the base, and the difference of the other two sides.



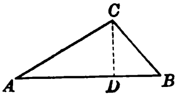
**Solution.** Draw $AB$ = the given base; at $A$ construct $\angle BAE$ = the given $\angle$; and on $AE$ take $AD$ = the given difference of sides.

Draw $BD$ and construct $\angle DBC = \angle BDC$, the side $BC$ meeting $AE$ as in $C$.

Then, $\triangle ABC$ is the $\triangle$ required.

**Proof.** Since, const., $\angle DBC = \angle BDC$, § 118, $BC = DC$;
hence, $AC - BC = AC - DC = AD$ = the given difference of sides.

Ex. **41**. Construct a triangle, having given the perpendicular from the vertex to the base, and the difference between each side and the adjacent segment of the base.



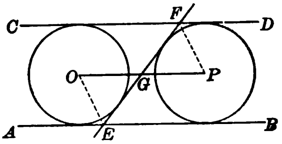
**Solution.** Draw $DC =$ the given perpendicular.

By Ex. 40 Rev., construct $\triangle ADC$ on $DC$ as base having $\angle ADC$ a rt. $\angle$, and $AC - AD =$ one of the given differences.

Similarly, construct $\triangle BDC$ on the other side of $DC$ as base having $\angle BDC$ a rt. $\angle$, and $BC - BD =$ the other given difference.

Then, $\triangle ABC$ is the $\triangle$ required.

Ex. **42**. If two circles are each tangent to two parallel lines and a transversal crossing them, the line of centers is equal to the segment of the transversal intercepted between the parallels.



**Data**: Ⓢ whose centers are $O$ and $P$ each tangent to the $\parallel$s $AB$ and $CD$, and to the transversal $EF$ that cuts $AB$ and $CD$ in $E$ and $F$, respectively.

**To prove** $OP = EF$.

**Proof.** Draw $OE$ and $PF$.

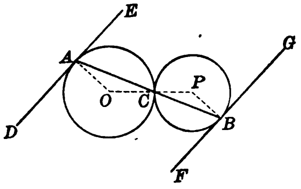
Since $O$ and $P$ are each equidistant from $AB$ and $CD$, Ex. 371, $OP \parallel AB$ and $CD$.

Then, § 73, $\angle GOE = \angle AEO$; but, Ex. 221, $\angle OEG = \angle AEO$;

$\therefore$ $\angle GOE = \angle OEG$, and, § 118, $OG = EG$; similarly, $GP = GF$;

hence, $OG + GP = EG + GF$; that is, $OP = EF$.

Ex. **43**. If through the point of contact of two circles which are tangent to each other externally any straight line is drawn terminated by the circumferences, the tangents at its extremities are parallel to each other.



**Data**: Ⓢ whose centers are $O$ and $P$ tangent externally at $C$, and line $AB$ through $C$ terminated in their circumferences respectively in $A$ and $B$; also $DE$ and $FG$, tangents at $A$ and $B$, respectively.

**To prove** $DE \parallel FG$.

**Proof.** Draw $OP$ and the radii $OA$ and $PB$.

Then, § 213, $OP$ passes through $C$.

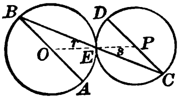
Since, Ax. 14, $OC = OA$, § 116, $\angle OAC = \angle OCA$;

similarly, $\angle PBC = \angle PCB$;

but, § 59, $\angle OCA = \angle PCB$; $\therefore \angle OAC = \angle PBC$;

hence, §§ 205, 54, $\angle EAB = \angle ABF$; $\therefore$ § 75, $DE \parallel FG$.

Ex. **44**. If two circles are tangent to each other externally and parallel diameters are drawn, the straight line joining the opposite extremities of these diameters will pass through the point of contact.



**Data**: Circles, whose centers are *O* and *P*, tangent externally at *E*; parallel diameters *AB* and *CD*; and the line *BC*.

**To prove** *BC* passes through *E*.

**Proof.** Draw the line of centers *OP*; § 213, *OP* passes through *E*. Draw *BE* and *EC*.

Since, § 73, $\angle BOE = \angle CPE$, § 54, $\angle B + \angle r = \angle C + \angle s$;

but, Ax. 14, $OB = OE$; $\therefore$ § 116, $\angle B = \angle r$; similarly, $\angle C = \angle s$;

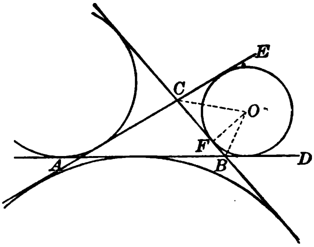
hence, $2\angle r = 2\angle s$; $\therefore \angle r = \angle s$.

Now, § 55, $\angle s + \angle OEC = 2$ rt. ∡;

hence, $\angle r + \angle OEC = 2$ rt. ∡;

consequently, § 58, *BE* and *EC* are in the same straight line; that is, *BC* passes through *E*.

Ex. **45**. Construct the three escribed circles of a given triangle.



**Solution.** Produce the sides *AB* and *AC* of the given △*ABC* to *D* and *E*, respectively.

Bisect ∡ *CBD* and *BCE* and produce the bisectors to intersect, as at *O*. Draw $OF \perp BC$.

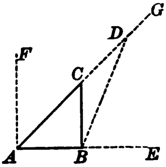
With *O* as center and *OF* as radius describe a ⊙.

Then, this ⊙ is an escribed ⊙ of △ *ABC*.

Similarly, the other escribed Ⓢ may be constructed.

**Proof.** § 134, *O* is equidistant from *BC* and *BD*, and *O* is equidistant from *BC* and *CE*; hence, *O* is equidistant from *BD*, *BC*, and *CE*; consequently, the ⊙ described with *O* as a center and *OF* as a radius touches *BD*, *BC*, and *CE*, and is an escribed ⊙ of △ *ABC*.

Ex. **46**. Construct an isosceles right triangle, having given the sum of the hypotenuse and one side.



**Solution.** Draw an indefinite line, as *AE*, and at *A* erect the perpendicular *AF*; draw *AG* to bisect $\angle FAE$, and on it take $AD =$ the given sum.

Construct $\angle ADB = \frac{1}{2} \angle DAE$, and produce the side to meet *AE* as at *B*; draw $BC \perp AE$ meeting *AD* as at *C*.

Then, △ *ABC* is the △ required.

**Proof.** $\angle BAD = \frac{1}{2}$ rt. $\angle$, and $\angle ADB = \frac{1}{4}$ rt. $\angle$;
hence, $\angle ABD = 2$ rt. $\angle$s $- (\frac{1}{2} + \frac{1}{4})$ rt. $\angle = \frac{5}{4}$ rt. $\angle$;
consequently, $\angle CBD = \frac{5}{4}$ rt. $\angle - 1$ rt. $\angle = \frac{1}{4}$ rt. $\angle$;
hence, § 118, $BC = CD$, and $AC + BC = AC + CD = AD$ = the given sum of the hypotenuse and one side.

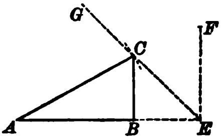
$\angle CAB = \frac{1}{2}$ rt. $\angle$; $\therefore$ § 111, $\angle ACB = \frac{1}{2}$ rt. $\angle$;
hence, § 118, $\triangle ABC$ is isosceles.

Ex. **47**. Construct a right triangle, having given the hypotenuse and the sum of the sides.

**Solution.** Draw $AE$ = the sum of the sides, and draw $EF \perp AE$.

Draw $EG$ to bisect $\angle AEF$; with $A$ as center and radius = the given hypotenuse, describe an arc cutting $EG$ as at $C$; draw $CB \perp AE$; and draw $AC$.

Then, $\triangle ABC$ is the $\triangle$ required.



**Proof.** § 71, $BC \parallel EF$; $\therefore$ § 73, $\angle BCE = \angle CEF$;
but, const., $\angle BEC = \angle CEF$;
hence, $\angle BCE = \angle BEC$, and, § 118, $BC = BE$.

Then, $AB + BC = AB + BE = AE$ = the given sum of sides.

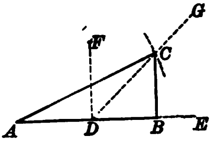
Ex. **48**. Construct a right triangle, having given the hypotenuse and the difference of the sides.

**Solution.** Draw the indefinite line $AE$, and on it take $AD$ = the given difference of sides; draw $DF \perp AE$; and draw $DG$ to bisect $\angle FDE$.

With $A$ as center and radius = the given hypotenuse, describe an arc intersecting $DG$ as at $C$.

Draw $CB \perp AE$, and draw $AC$.

Then, $\triangle ABC$ is the $\triangle$ required.

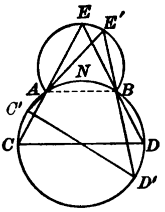


**Proof.** § 71, $BC \parallel DF$; $\therefore \angle BCD = \angle CDF$;
but, const., $\angle BDC = \angle CDF$;
hence, $\angle BCD = \angle BDC$, and, § 118, $BC = BD$.

Then, $AB - BC = AB - BD = AD$ = the given difference of sides.

Ex. **49**. $A$ and $B$ are two fixed points on the circumference of a circle, and $CD$ is any diameter. What is the locus of the intersection of $CA$ and $DB$?

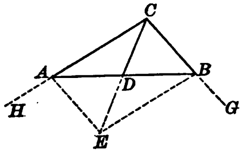
**Solution.** Since for any positions of $CD$, as $CD$ and $C'D'$, § 234, the $\angle$s $E$ and $E'$ at the intersections of $CA$, $DB$, and $C'A$, $D'B$ are measured by $\frac{1}{2}$ (semicircum. − arc $ANB$) or its equal, these $\angle$s are equal, and the required locus is the locus of the vertices of the vert. $\angle$s of $\triangle$s on the same base and having equal vert. $\angle$s; that is, Ex. 268, Cor., the locus is the arc $AEE'B$ of the segment containing $\angle E$, constructed on the chord $AB$.



Ex. **50**. Construct a triangle, having given a median and the two angles into which the angle is divided by that median.

**Solution.** Draw $CD =$ the given median, and construct $\angle$s $DCH$ and $DCG =$ resp. the given $\angle$s.

Produce $CD$ to $E$, making $DE = CD$, and draw $EB \parallel HC$ and $EA \parallel GC$, meeting $GC$ and $HC$ respectively as in $B$ and $A$. Draw $AB$.



Then, $\triangle ABC$ is the $\triangle$ required.

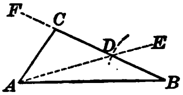
**Proof.** § 140, $AEBC$ is ▱, and, § 154, its diagonals $AB$ and $EC$ bisect each other, and since, const., $D$ is the middle point of $EC$, it must be the middle point of $AB$; consequently, $CD$ is a median of $\triangle ABC$.

Ex. **51**. Construct a triangle, having given the base, the difference between the sides, and the difference between the angles at the base.

**Solution.** Draw $AB =$ the given base, and at $A$ construct $\angle BAE =$ one half the given difference of the base $\angle$s.

With $B$ as a center and a radius $=$ the given difference of sides, describe an arc cutting $AE$, as at $D$.

Through $D$ draw $BF$ and construct $\angle DAC = \angle ADF$, the side $AC$ meeting $BF$ as at $C$.



Then, $\triangle ABC$ is the $\triangle$ required.

**Proof.** § 118, $AC = DC$; hence, $BC - AC = BC - DC = BD =$ the given difference of sides.

$\angle CAB + \angle B =$ sup. of $\angle ACB$, and $\angle CAD + \angle ADC =$ sup. of $\angle ACB$; hence, $\angle CAD + \angle ADC = \angle CAB + \angle B$, and since $\angle CAD = \angle ADC$,

$\angle CAD = \frac{1}{2}(\angle CAB + \angle B)$; but, § 115, $\angle ADC = \angle DAB + \angle B$;

$\therefore \angle CAD = \angle DAB + \angle B$; hence, $\angle DAB + \angle B = \frac{1}{2}(\angle CAB + \angle B)$;

whence, $\angle DAB = \frac{1}{2}(\angle CAB + \angle B) - \angle B = \frac{1}{2}(\angle CAB - \angle B)$;

but, const., $\angle DAB = \frac{1}{2}$ the given difference of base $\angle$s;

hence, $\frac{1}{2}(\angle CAB - \angle B) = \frac{1}{2}$ the given difference of base $\angle$s;

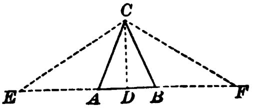
and $\angle CAB - \angle B =$ the given difference of base $\angle$s.

**Sch.** If the given difference of sides is *less* than the distance from $B$ to $AE$, the solution is impossible; if *greater*, there are two solutions; and if *equal* to the distance from $B$ to $AE$, there is only one solution.

Ex. **52**. Construct an isosceles triangle, having given the perimeter and altitude.

**Solution.** Draw $EF =$ the given perimeter, and at its middle point $D$, draw $DC =$ the given altitude.

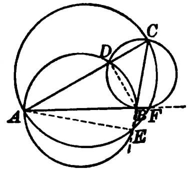
Draw $EC$ and $FC$; construct $\angle ECA = \angle E$, and $\angle FCB = \angle F$, the sides $CA$ and $CB$, meeting $EF$ in $A$ and $B$ respectively.



Then, $\triangle ABC$ **is the** $\triangle$ **required.**

**Proof.** Const., § 118, $AC = AE$, and $BC = BF$;
hence, $AC + AB + BC = EA + AB + BF = EF =$ the given perimeter.
§ 103, $EC = FC$; $\therefore$ in $\triangle EFC$, § 116, $\angle E = \angle F$;
consequently, $\angle ECA = \angle BCF$; hence, § 102, $\triangle ACE = \triangle BCF$,
and $AC = BC$; that is, $\triangle ABC$ is isosceles.

**Ex. 53.** The circles described on two sides of a triangle as diameters intersect on the third side, or the third side produced.



**Data:** $\triangle ABC$ and the Ⓢ described on its sides as diameters and intersecting in the points $D$, $E$, and $F$.

**To prove** $D$, $E$, and $F$ on the sides $AC$, $BC$, and $AB$ respectively or on those sides produced.

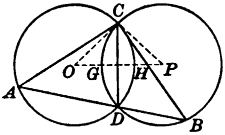
**Proof.** (1) Draw $AD$, $BD$, and $CD$.

Then, § 227, $\angle ADB$ is a rt. $\angle$, and $\angle BDC$ is a rt. $\angle$; consequently, § 58, $AD$ and $DC$ form a straight line; but, $AC$ is a straight line; $\therefore$ Ax. 11, $ADC$ and $AC$ coincide; hence, $D$ is on $AC$.

(2) Draw $AE$, $BE$, $CE$; then, § 227, $\angle AEB$ is a rt. $\angle$, and $\angle AEC$ is a rt. $\angle$; that is, $BE$ and $CE \perp AE$ at $E$; consequently, since, § 51, there can be only one $\perp$ to $AE$, at $E$, $BE$ and $CE$ are in the same straight line; that is, $E$ is on $CB$ produced.

Similarly it may be shown that $F$ is on $AB$ produced.

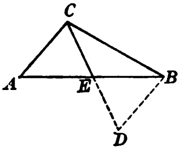
**Ex. 54.** $ABC$ is a triangle having $AC$ equal to $BC$; $D$ is any point in $AB$. Prove that the circles circumscribed about triangles $ADC$ and $DBC$ are equal.



**Proof.** Let $O$ and $P$ be the centers respectively of the circumscribed Ⓢ. Draw $OP$, $OC$, and $PC$.

§ 212, $OP \perp CD$ at its middle point;
$\therefore$ § 200, $OP$ bisects arcs $CHD$ and $CGD$;
hence, §§ 224, 225, $\angle O = \angle A$, each being measured by $\frac{1}{2}$ arc $CHD$,
and $\angle P = \angle B$, each being measured by $\frac{1}{2}$ arc $CGD$;
but, § 116, $\angle A = \angle B$; $\therefore$ $\angle O = \angle P$; hence, § 118, $OC = PC$;
but $OC$ and $PC$ are the radii respectively of the Ⓢ circumscribed about $\triangle$ $ADC$ and $DBC$ respectively; $\therefore$ Ax. 16, the Ⓢ are equal.

**Ex. 55.** Construct a triangle, having given two sides and the median to the third side.



**Solution.** Select the point $C$ as the vertex where the three given lines intersect and draw $CB =$ one of the given sides.

Construct the $\triangle BCD$ with $CD =$ double the given median and $BD =$ the other given side.

Find $E$ the middle point of $CD$; draw $BE$ and produce it to $A$ making $EA = BE$. Draw $AC$.

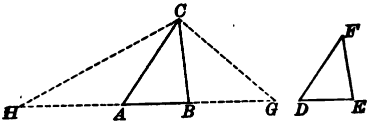
Then, $\triangle ABC$ is the $\triangle$ required.

**Proof.** Const., $AE = BE$, $CE = DE$, and, § 59, $\angle AEC = \angle BED$;

$\therefore$ § 100, $\triangle AEC = \triangle BED$, and $AC = BD =$ one of the given sides,

const., $BC =$ the other given side, and $CE$ is a median of $\triangle ABC$ equal to the given median.

**Ex. 56.** Construct a triangle, having given its perimeter, and having its angles equal to the angles of a given triangle.



**Solution.** Draw $HG =$ the given perimeter; construct $\angle H = \frac{1}{2} \angle D$ of the given $\triangle DEF$, and $\angle G = \frac{1}{2} \angle E$, producing their sides to meet, as at $C$.

Construct $\angle HCA = \angle H$, and $\angle GCB = \angle G$, $CA$ and $CB$ meeting $HG$ in $A$ and $B$.

Then, $\triangle ABC$ is the $\triangle$ required.

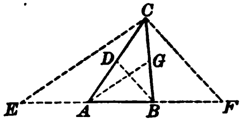
**Proof.** Const., § 118, $AC = HA$, and $BC = BG$;

hence, $AC + AB + BC = HA + AB + BG = HG =$ the given perimeter.

$\angle BAC = \angle H + \angle ACH = 2 \angle H = \angle D$; similarly, $\angle ABC = \angle E$,

and, § 113, $\angle ACB = \angle F$; hence, $\triangle ABC$ is the $\triangle$ required.

**Ex. 57.** Construct a triangle, having given one side and the medians to the other sides.



**Solution.** Draw $EF = 3$ times the given side and on it take $EA$, $AB$, and $BF$ each equal the given side.

Construct $\triangle EFC$ with $EC = 2$ times one of the given medians and $FC = 2$ times the other given median. Draw $AC$ and $BC$.

Then, $\triangle ABC$ is the triangle required.

**Proof.** Draw $AG \parallel EC$, meeting $BC$ in $G$, and $BD \parallel FC$, meeting $AC$ in $D$.

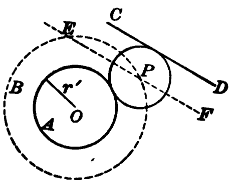
Then, § 158, $AG$ bisects $BC$, and $AG = \frac{1}{2} EC =$ one of the given medians;

that is, $AG$ is a median of $\triangle ABC$ and equal to one of the given medians;

similarly, $BD$ is a median equal the other given median;

hence, $\triangle ABC$ is the $\triangle$ required.

**Ex. 58.** Construct a circle of given radius to touch a given circle and a given straight line. How many such circles may there be?



**Solution.** Draw the $\odot B$ concentric with the given $\odot A$ having its radius $= r + r'$, the sum of the given radius, and the radius of the given $\odot$.

Draw $EF \parallel CD$, the given line, and at a distance from it equal to $r$, cutting the circumference of $\odot B$, as in $P$ and $E$.

With $P$ as center and radius $= r$, describe a $\odot$.

Then, this $\odot$ is the $\odot$ required.

**Proof.** Since, const., $P$ is on $EF$ and the circum. of $\odot B$, it is at the distance $r$ from $CD$ and the circum. of $\odot A$.

Hence, the $\odot$ whose center is $P$ and radius $r$ touches $\odot A$ and line $CD$.

**Sch.** Since there may be two lines parallel to $CD$ at the distance $r$ from it, and $\odot B$ may be described with a radius equal to the difference between $r$ and $r'$, there are eight possible solutions of this problem.

Let $x$ denote the distance from center $O$ to $CD$.

Then, if $r' > r$, and $x > r' + 2r$, all the solutions are impossible.

If $r' > r$, and $x = r' + 2r$, two solutions coincide, and the rest are impossible.

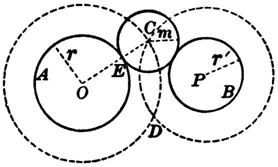
If $r' > r$, and $x$ is between $r'$ and $r' + 2r$, only two solutions are possible.

If $r' > r$, and $x = r'$, there are two pairs of coincident solutions, and the rest are impossible.

If $r' > r$, and $x < r'$, six solutions are possible.

If $r' > 2r$, and $x < r' - 2r$, all eight solutions are possible.

**Ex. 59.** Construct a circle of given radius which shall be tangent to two given circles. How many solutions may there be?



**Solution.** Let $A$ and $B$ be the given $\circledS$ whose centers are respectively $O$ and $P$ and radii $r$ and $r'$, and let $m$ be the given radius.

With $O$ as center and radius equal the sum (or difference) of $m$ and $r$, describe a $\odot$; with $P$ as center and radius equal the sum (or difference) of $m$ and $r'$ describe another $\odot$ intersecting the first, as in $C$ and $D$.

With $C$ as center and radius $= m$, describe a $\odot$.

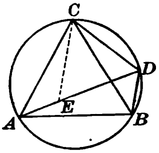
Then, this $\odot$ is the $\odot$ required.

**Proof.** Draw $OC$. $OC = r + m$, and $CE = m$; hence, the $\odot$ whose center is $C$ touches the circum. of $A$.

Similarly, this $\odot$ touches the circum. of $B$.

**Sch.** With $D$ as center and radius $= m$, another $\odot$ fulfilling the conditions may be constructed, and in general there are eight possible solutions, according as the sum or difference, or the sum and difference, of radii are used in describing the auxiliary $\circledS$.

Ex. **60**. If an equilateral triangle is inscribed in a circle and from any point in the circumference lines are drawn to the vertices, the longest of these lines is equal to the sum of the other two.



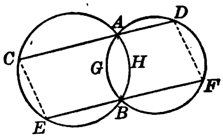
**Data**: Equilateral $\triangle ABC$ inscribed in a $\odot$; $D$ any point in the circumference; and the lines $AD$, $BD$, and $CD$ of which $AD$ is the longest.

**To prove** $AD = BD + CD$.

**Proof.** Take $DE = CD$ and draw $CE$.

§ 225, $\angle ADC = \angle ABC$, each being measured by $\frac{1}{2}$ arc $AC$,
and, since, Ex. 74, $\angle ABC = 60°$, $\angle ADC = 60°$;
but, § 116, $\angle ECD = \angle CED = \frac{1}{2}(180° - 60°) = 60°$;
hence, § 119, $\triangle EDC$ is equilateral, and in $\triangle$s $AEC$ and $BDC$,
$EC = DC$, $AC = BC$, and, Ax. 3, $\angle ACD - \angle ECD = \angle ACD - \angle ACB$,
or $\angle ACE = \angle BCD$;
$\therefore$ § 100, $\triangle AEC = \triangle BDC$, and $AE = BD$; but, const., $ED = CD$;
hence, $AE + ED = BD + CD$; that is, $AD = BD + CD$.

Ex. **61**. If two circles intersect each other, two parallel lines passing through the points of intersection and terminated by the exterior arcs are equal.



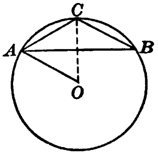
**Data**: Two $\circledS$ that intersect at $A$ and $B$; and the $\parallel$s $CD$ and $EF$ through $A$ and $B$ respectively and terminated by the external arcs.

**To prove** $CD = EF$.

**Proof.** Draw $CE$ and $DF$.

Ex. 253, $CE \parallel DF$, and, data, $CD \parallel EF$; $\therefore$ § 140, $EFDC$ is a parallelogram, and, § 153, $CD = EF$.

Ex. **62**. An isosceles triangle has its vertical angle equal to an exterior angle of an equilateral triangle. Prove that the radius of the circumscribed circle is equal to one of the equal sides of the isosceles triangle.



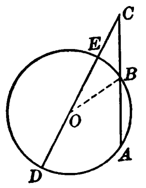
**Data**: Isos. $\triangle ABC$ whose vert. $\angle C =$ ext. $\angle$ of an equilateral $\triangle$; and the $\odot$ whose center is $O$ and radius $OA$, circumscribed about $\triangle ABC$.

**To prove** $OA = AC$.

**Proof.** Draw $OC$.

Ax. 14, § 90, $O$ and $C$ are each equidistant from $A$ and $B$; $\therefore$ § 106, $OC \perp AB$ at its middle point;
hence, § 121, $OC$ bisects $\angle ACB$, and $\angle ACO = \frac{1}{2}(180° - 60°) = 60°$;
but, Ax. 14, $OA = OC$; $\therefore$ § 116, $\angle OAC = \angle ACO = 60°$;
hence, $\angle AOC = 60°$, and, § 119, $\triangle AOC$ is equilateral;
that is, $OA = AC$.

**Ex. 63.** If a chord of a circle is extended by a length equal to the radius, and from the extremity a secant is drawn through the center of the circle, the length of the greater included arc is three times the length of the less.



**Data:** ⊙ whose center is $O$; chord $AB$ produced to $C$ so that $BC =$ the radius; and the secant from $C$ through $O$ cutting the circumference in $E$ and meeting it in $D$.

**To prove** arc $AD = 3$ times arc $BE$.

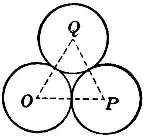
**Proof.** Draw $OB$.

Data, $BC = OB$; $\therefore$ § 116, $\angle C = \angle BOC$; but $\angle BOC$ is measured by arc $BE$; $\therefore$ $\angle C$ is measured by arc $BE$;

but, § 234, $\angle C$ is measured by $\frac{1}{2}$(arc $AD$ − arc $BE$);

hence, arc $BE = \frac{1}{2}$ arc $AD - \frac{1}{2}$ arc $BE$; whence, arc $AD = 3$ arc $BE$.

**Ex. 64.** Construct three circles having equal diameters and being tangent to each other.



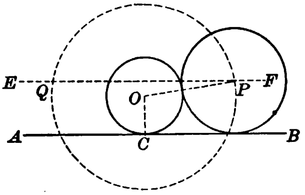
**Solution.** Draw $OP =$ the given diameter.

With $O$ and $P$ as centers and radii $= \frac{1}{2}\,OP$ describe Ⓢ and, Ex. 59, Rev., construct a ⊙ whose center is $Q$ and radius $\frac{1}{2}\,OP$, tangent to these two Ⓢ.

Then, the Ⓢ whose centers are $O$, $P$, and $Q$ are the Ⓢ required.

**Proof.** Proceed as in Ex. 342, to prove the Ⓢ whose centers are $O$ and $P$ tangent to each other, and, by const., the circle whose center is $Q$ tangent to each of these.

**Ex. 65.** Construct two circles of given radii to touch each other and a given straight line on the same side of it.



**Solution.** Let $r$ and $r'$ be the given radii.

At any point as $C$ in $AB$, the given line, draw $CO \perp AB$ and equal to $r$; with $O$ as center and radius $r$ describe a ⊙.

On the same side of $AB$ as $O$ draw $EF \parallel AB$ and at the distance $r'$ from $AB$.

With $O$ as center and radius equal the sum (or difference) of $r$ and $r'$ describe a ⊙ cutting $EF$ as in $P$ and $Q$.

With $P$ as center and radius $r'$ describe a ⊙.

Then, the Ⓢ whose centers are $O$ and $P$ are the Ⓢ required.

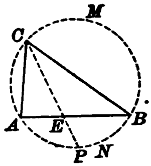
**Proof.** § 205, the ⊙ whose center is $O$ touches $AB$, and since $P$ is at the distance $r'$ from $AB$ and the ⊙ whose center is $O$, the ⊙ whose center is $P$ and radius $r'$ touches $AB$ and the ⊙ whose center is $O$, since $OP = r + r'$.

**Sch.** $Q$ is the center of another ⊙ fulfilling the conditions, and theoretically there are four solutions.

The ⊙ whose radius is $r + r$ will always give two possible distinct solutions.

The ⊙ whose radius is the difference between $r$ and $r'$ gives two coincident solutions.

Ex. **66**. Construct a triangle, having given the base, the vertical angle, and the point at which the base is cut by the bisector of the vertical angle.



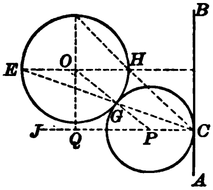
**Solution.** On the given base $AB$ describe a segment $AMB$ that will contain the given $\angle$; and find the middle point $P$ of arc $ANB$.

Draw a line from $P$ to the given point $E$ in $AB$, and produce it to meet arc $AMB$ as at $C$. Draw $AC$ and $BC$.

Then, $\triangle ABC$ is the $\triangle$ required.

**Proof.** Since, const., arc $AP =$ arc $BP$, § 225, $\angle ACP = \angle BCP$; that is, $CP$ is the bisector of $\angle ACB$ and cuts the base at $E$.

Ex. **67**. Construct a circle to touch a given circle, and also to touch a given straight line at a given point.



**Solution.** At the given point $C$ in the given line $AB$ draw $CJ \perp AB$.

Draw the diameter $EH$ of the given ⊙, whose center is $O$, perpendicular to $AB$.

Join $C$ with one extremity, as $E$, of diameter $EH$, by the line $CE$, cutting the circumference of the given ⊙ as at $G$.

Draw $OG$ and produce it to cut $CJ$ as at $P$.

With $P$ as center and $PC$ as radius describe a ⊙.

Then, this ⊙ is the ⊙ required.

**Proof.** Ax. 14, $OE = OG$; §§ 116, 59, $\angle OEG = \angle OGE = \angle PGC$, and, §§ 71, 73, $\angle OEG = \angle GCP$; hence, $\angle PGC = \angle GCP$, and $PG = PC$, and the ⊚ may be proved tangent, as in Ex. 342; also, since $AB \perp PC$ at $C$, § 205, the ⊙ whose center is $P$ is tangent to $AB$.

**Sch.** By drawing $CH$ instead of $CE$ and producing it to meet the circumference of the given ⊙, another solution may be obtained, the center being at $Q$.

If $AB$ is without the given ⊙, the ⊙ resulting from $CE$ has external contact, and the ⊙ resulting from $CH$ has internal contact, the given ⊙ being within the other.

If $AB$ is tangent to the given ⊙, the ⊙ derived from $CE$ has external contact, and the other solution is impossible.

If $AB$ cuts the given ⊙, both constructed ⊚ are tangent externally, or both internally, according as the point $C$ is without or within the given ⊙.

Ex. **68**. Construct a circle to touch a given straight line, and to touch a given circle at a given point.

**Solution.** (See Fig. for Ex. 67.) Draw the diameter *EH* of the given ⊙ whose center is *O*, perpendicular to the given line *AB*.

From one extremity of *EH*, as *E*, draw a line to *G*, the given point, and produce it to meet *AB* as at *C*; draw *CJ* ⊥ *AB*; draw *OG* and produce it to meet *CJ* as at *P*.

With *P* as center and radius *PC* describe a ⊙.

Then, this ⊙ is the ⊙ required.

**Proof.** Proceed as in Ex. 67, Rev.

**Sch.** There are two possible solutions having external or internal contact under the same conditions as in Ex. 67, Rev., except in the case when *AB* cuts the given ⊙; then, one of the resulting Ⓢ has internal contact, and is within the given ⊙, while the other is tangent externally.

**Ex. 69.** Construct a circle to touch a given circle, have its center in a given line, and pass through a given point in that line.

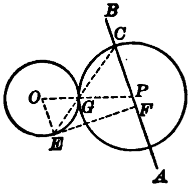
**Solution.** Ex. 338, draw *EF* tangent to the given ⊙, whose center is *O*, and perpendicular to the given line *AB*.

From *E*, the point of tangency, draw *EC* to *C*, the given point in *AB*, and cutting the circumference as in *G*.

Draw *OG* and produce it to meet *AB*, as in *P*.

With *P* as center and radius *PG* describe a ⊙.

Then, this ⊙ is the ⊙ required.



**Proof.** Draw *OE*; then, §§ 205, 71, *OE* ∥ *AB*.

For the rest of proof, proceed as in Ex. 67, Rev.

**Sch.** Since another tangent can be drawn to the given ⊙ perpendicular to *AB*, there are two solutions of the problem.

**Ex. 70.** If $a : b = c : d$, prove that

$$a^2 + ab + b^2 : a^2 - ab + b^2 = c^2 + cd + d^2 : c^2 - cd + d^2.$$

**Proof.** § 288, $a^3 : b^3 = c^3 + d^3;$

then, § 278, $a^3 + b^3 : a^3 - b^3 = c^3 + d^3 : c^3 - d^3,$

or, § 275, $a^3 - b^3 : a^3 + b^3 = c^3 - d^3 : c^3 + d^3;$

that is, $$\frac{(a-b)(a^2+ab+b^2)}{(a+b)(a^2-ab+b^2)} = \frac{(c-d)(c^2+cd+d^2)}{(c+d)(c^2-cd+d^2)}; \qquad (1)$$

but, data, § 278, $a + b : a - b = c + d : c - d,$

or $$\frac{a+b}{a-b} = \frac{c+d}{c-d};$$

then, multiplying the first member of (1) by $\frac{a+b}{a-b}$, and the second member by $\frac{c+d}{c-d}$,

$$\frac{a^2+ab+b^2}{a^2-ab+b^2} = \frac{c^2+cd+d^2}{c^2-cd+d^2};$$

that is, $a^2 + ab + b^2 : a^2 - ab + b^2 = c^2 + cd + d^2 : c^2 - cd + d^2.$

Ex. **71**. Given three lines $a$, $b$, and $c$. Construct $x = \frac{bc}{a}$.

**Solution.** $ax = bc$; $\therefore$ § 273, $a : b = c : x$.

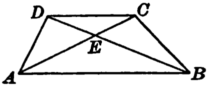
Then, as in § 317, construct $x$, a fourth proportional to $a$, $b$, and $c$.

Ex. **72**. Construct $x$, having given $\frac{4}{x} = \frac{x}{9}$.

**Solution.** $x^2 = 4 \cdot 9$; $\therefore$ § 273, $4 : x = x : 9$.

Then, as in § 319, construct $x$, a mean proportional between 4 and 9.

Ex. **73**. The diagonals of a trapezoid divide each other into segments which are proportional.

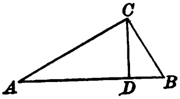


**Data** : Trapezoid $ABCD$ and its diagonals $AC$ and $BD$ intersecting in $E$.

**To prove** $AE : CE = BE : DE$.

**Proof.** Ex. 559, $\triangle$ $ABE$ and $CDE$ are similar;
hence, § 299, $AE : CE = BE : DE$.

Ex. **74**. If one side of a right triangle is double the other, the perpendicular from the vertex upon the hypotenuse divides the hypotenuse into parts which are in the ratio of 1 to 4.



**Data**: Rt. $\triangle ABC$ in which $AC = 2\,BC$; also $CD \perp AB$, the hypotenuse.

**To prove** $BD : AD = 1 : 4$.

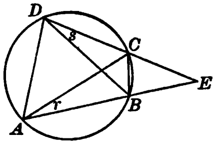
**Proof.** § 313, $\overline{BC}^2 = AB \times BD$, and $\overline{AC}^2 = AB \times AD$;

then, $\overline{BC}^2 : \overline{AC}^2 = AB \times BD : AB \times AD = BD : AD$;

but, data, $BC : AC = 1 : 2$; $\therefore$ $\overline{BC}^2 : \overline{AC}^2 = 1 : 4$;

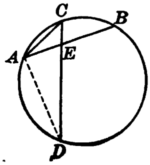
hence, $BD : AD = 1 : 4$.

Ex. **75**. $ABCD$ is an inscribed quadrilateral. The sides $AB$ and $DC$ are produced to meet at $E$. Prove triangles $ACE$ and $BDE$ similar.



**Proof.** § 225, $\angle r = \angle s$, each being measured by $\frac{1}{2}$ arc $BC$, and $\angle E$ is common; hence, § 301, $\triangle$ $ACE$ and $BDE$ are similar.

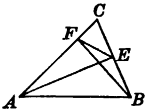
Ex. **76**. If $AB$ is a chord of a circle, and $CD$ is any chord drawn from the middle point $C$ of the arc $AB$ cutting the chord $AB$ at $E$, prove that the chord $AC$ is a mean proportional between $CE$ and $CD$.



**Proof.** Draw $AD$.

Then, in $\triangle$ $AEC$ and $ADC$, $\angle C$ is common, and, § 225, $\angle CAE = \angle D$, being measured respectively by $\frac{1}{2}$ the equal arcs $BC$ and $AC$; hence, § 301, $\triangle$ $AEC$ and $ADC$ are similar, and, § 299, $CE : AC = AC : CD$.

**Ex. 77.** If perpendiculars are drawn from two vertices of a triangle to the opposite sides, the triangle cut off by the line joining the feet of the perpendiculars is similar to the original triangle.



**Data**: $\triangle ABC$; $AE \perp BC$; $BF \perp AC$; and line $EF$.

**To prove** $\triangle$s $ABC$ and $EFC$ similar.

**Proof.** In rt. $\triangle$s $AEC$ and $BFC$, $\angle C$ is common;

$\therefore$ § 302, $\triangle$s $AEC$ and $BFC$ are similar;

hence, in $\triangle$s $ABC$ and $EFC$, $AC : BC = EC : FC$, and $\angle C$ is common;

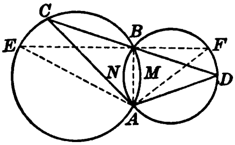
$\therefore$ § 306, $\triangle$s $ABC$ and $EFC$ are similar.

**Ex. 78.** $AB$ is the hypotenuse of the right triangle $ABC$. If perpendiculars are drawn to $AB$ at $A$ and $B$, meeting $BC$ produced at $E$ and $AC$ produced at $D$, the triangles $ACE$ and $BCD$ are similar.

**Proof.** § 71, $AE \parallel BD$; $\therefore$ § 73, $\angle EAC = \angle CDB$;

hence, § 302, $\triangle$s $ACE$ and $BCD$ are similar.

**Ex. 79.** If two circles intersect in the points $A$ and $B$, and a secant through $B$ cuts the circumferences in $C$ and $D$, respectively, the straight lines $AC$ and $AD$ are in the same ratio as the diameters of the circles.



**Proof.** Draw the diameters from $A$ meeting the circumferences in $E$ and $F$, respectively; draw $AB$, $EB$, and $FB$.

Then, § 227, $\triangle$s $ABE$ and $ABF$ are rt. $\triangle$s;

$\therefore$ § 58, $EBF$ is a straight line.

Then, in $\triangle$s $ADC$ and $AFE$,

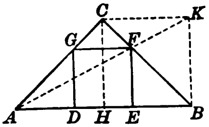
§ 225, $\angle C = \angle E$ (each being measured by $\frac{1}{2}$ arc $AMB$),

and $\angle D = \angle F$ (each being measured by $\frac{1}{2}$ arc $ANB$);

hence, § 301, $\triangle$s $ADC$ and $AFE$ are similar,

and, § 299, $AC : AD = AE : AF$.

**Ex. 80.** Inscribe a square in a given right isosceles triangle.



**Solution.** Draw $CH$, the altitude of the given $\triangle ABC$.

On $HB$ or $HB$ produced construct the square $HBKC$ with $CH$ as one side.

Draw $AK$ cutting $BC$ as in $F$; draw $FG \parallel AB$, and draw $FE$ and $GD \perp AB$.

Then, $DEFG$ is the square required.

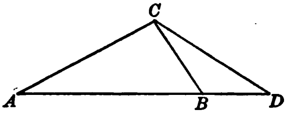
**Proof.** § 71, $EF \parallel BK$; $\therefore$ § 301, $\triangle$s $AEF$ and $ABK$ are similar;

hence, § 299, $EF : BK = AF : AK$; similarly, $GF : CK = AF : AK$;

$\therefore$ $EF : BK = GF : CK$; but, § 143, $BK = CK$;

consequently, § 272, $EF = GF$, and since, const., $DEFG$ is a rectangle, § 153, all of its sides are equal, and, § 143, $DEFG$ is the required square.

Ex. **81.** From the obtuse angle of a triangle draw a line to the base, which shall be a mean proportional between the external segments into which it divides the base.

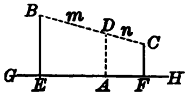


**Solution.** From $C$, the vertex of the obtuse $\angle$ of the given $\triangle ABC$, draw $CD$ to meet the base $AB$ produced as at $D$, making $\angle ACD = \angle CBD$.

Then, $CD$ is a mean proportional between the external segments of the base.

**Proof.** In the $\triangle$s $ADC$ and $BDC$, $\angle D$ is common, and, const., $\angle ACD = \angle CBD$; $\therefore$ § 301, $\triangle$s $ADC$ and $BDC$ are similar; hence, § 299, $AD : CD = CD : BD$.

Ex. **82.** Through a given point draw a straight line, so that the parts of it intercepted between that point and perpendiculars drawn to it from two other given points may have a given ratio.



**Solution.** Let $A$ be the first given point; $B$ and $C$ the other given points; and $m : n$ the given ratio.

Draw the line $BC$ and divide it, as at $D$, into segments having the ratio $m : n$.

Draw $AD$, and at $A$ draw $GH \perp AD$.

Then, $GH$ is the required line.

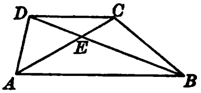
**Proof.** Draw $BE$ and $CF \perp GH$.

If $BC \parallel GH$, § 151, $EA = BD$, and $AF = DC$;

then, $EA : AF = BD : DC = m : n$.

If $BC$ and $GH$ are not parallel, produce them to intersect; then, from similar $\triangle$s it may be shown that $EA : AF = BD : DC = m : n$.

Ex. **83.** Show that the diagonals of a trapezoid, one of whose bases is double the other, cut each other at a point of trisection.



**Data**: Trapezoid $ABCD$ in which $AB = 2\,DC$, and whose diagonals $AC$ and $BD$ cut each other at $E$.

**To prove** $E$ a point of trisection of $AC$ and $BD$.

**Proof.** Ex. 559, $\triangle$s $DCE$ and $ABC$ are similar; hence, § 299, $DC : AB = EC : AE$; but, data, $DC : AB = 1 : 2$; $\therefore$ $EC : AE = 1 : 2$; whence, $EC = \frac{1}{2} AE = \frac{1}{3} AC$; similarly, $ED = \frac{1}{3} BD$; that is, $E$ is a point of trisection of $AC$ and $BD$.

Ex. **84.** A tangent to a circle at the point $A$ intersects two parallel tangents whose points of contact are $D$ and $E$, in $B$ and $C$ respectively. $BE$ and $CD$ intersect at $F$. Prove that the line $AF$ is parallel to the tangents $BD$ and $CE$.

**Proof.** § 301, $\triangle$s $CFE$ and $BFD$ are similar;

∴ § 299, $CF : FD = CE : BD$; but, § 209, $CE = CA$, and $BD = AB$;

∴ substituting, $CF : FD = CA : AB$; hence, § 291, $AF \parallel BD$,

and, § 80, $AF \parallel CE$.

Ex. **85**. The angle $C$ of the triangle $ABC$ is bisected by $CD$, which cuts the base $AB$ at $D$; $O$ is the middle point of $AB$. Prove that $OD$ has the same ratio to $OA$ that the difference of the sides has to their sum.

**Proof.** Let $AC$ be greater than $BC$. § 292, $AD : DB = AC : BC$; (1)

hence, § 277, $AD\text{-}DB : DB = AC\text{-}BC : BC$;

that is, § 274, $2\,OD : AC\text{-}BC = DB : BC$. (2)

From (1), § 276, $AD + DB : DB = AC + BC : BC$;

that is, § 274, $2\,OB : AC + BC = DB : BC$. (3)

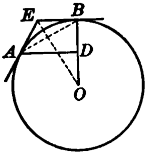
Hence, from (2) and (3), Ax. 1, $2\,OD : AC\text{-}BC = 2\,OB : AC + BC$;

∴ § 274, $2\,OD : 2\,OB = AC\text{-}BC : AC + BC$;

that is, § 283, $OD : OB = AC\text{-}BC : AC + BC$;

that is, $OD : OA = AC\text{-}BC : AC + BC$.

Ex. **86**. $A$ and $B$ are two points on the circumference of a circle of which $O$ is the center; tangents at $A$ and $B$ meet at $E$; and from $A$ the line $AD$ is drawn perpendicular to $OB$. Prove $BE : BO = BD : AD$.



**Proof.** Draw $AB$ and $EO$.

§ 205, $EB \perp OB$; ∴ § 71, $EB \parallel AD$

and, § 73, $\angle BAD = \angle ABE$,

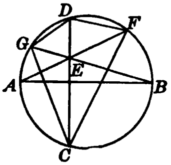
also, since, Ex. 221, $EO$ bisects arc $AB$,

§§ 224, 231, $\angle EOB = \angle ABE$ (each being measured by $\frac{1}{2}$ arc $AB$); hence, $\angle EOB = \angle BAD$;

hence, § 302, $\triangle$s $OBE$ and $ADB$ are similar,

and, § 299, $BE : BO = BD : AD$.

Ex. **87**. $AB$ is a diameter of a circle, $CD$ is a chord at right angles to it, and $E$ is any point in $CD$; $AE$ and $BE$ are drawn and produced to cut the circumference in $F$ and $G$ respectively. Prove that $CFDG$ has any two of its adjacent sides in the same ratio as the remaining two.



**Proof.** Since $AB \perp CD$, § 209, arc $AD =$ arc $AC$;

∴ § 225, $\angle AFD = \angle AFC$;

hence, § 292, $DE : CE = DF : CF$.

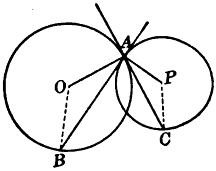
Similarly, $GE$ bisects $\angle CGD$,

and $DE : CE = DG : CG$.

Hence, $DF : CF = DG : CG$, and, § 274, $DF : DG = CF : CG$.

**Ex. 88.** Two circles whose centers are $O$ and $P$ intersect in $A$, and the tangent to each at $A$ meets the circumference of the other in $C$ and $B$ respectively. Prove that $AB : AC = OA : PA$.

**Proof.** Draw $OB$ and $PC$.



§ 205, $\triangle$s $OAC$ and $PAB$ are rt. $\angle$s;

$\therefore$ $\angle$s $OAB$ and $PAC$ are each the comp. of $\angle CAB$;

hence, § 54, $\angle AOB = \angle PAC$;

since, Ax. 14, $OA = OB$,

§ 116, $\angle B = \angle OAB$; similarly, $\angle C = \angle PAC$;

$\therefore$ $\angle B = \angle C$; hence, § 301, $\triangle$s $OBA$ and $ACP$ are similar;

and, § 299, $AB : AC = OA : PA$.

Ex. **89**. $O$ is the center of the circle inscribed in the triangle $ABC$; $AO$ meets $BC$ in $D$. Prove that $AO : DO = AB + AC : BC$.

**Proof.** Ex. 221, $AD$ bisects $\angle BAC$;

hence, § 292, $BD : DC = AB : AC$;

$\therefore$ § 276, $BD + DC : DC = AB + AC : AC$,

or, § 274, $BD + DC : AB + AC = DC : AC$;

Draw $CO$; then, $CO$ bisects $\angle ACD$, and $DO : AO = DC : AC$.

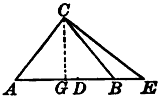
Hence, $DO : AO = BD + DC : AB + AC$;

that is, $DO : AO = BC : AB + AC$,

or, § 275, $AO : DO = AB + AC : BC$.

Ex. **90**. $ABC$ is an isosceles triangle; the perpendicular to $AC$ at $C$ meets the base $AB$, or the base produced, at $E$; $D$ is the middle point of $AE$. Prove that $AC$ is a mean proportional between $AB$ and $AD$.

**Proof.** Draw $CG \perp AE$.



Then, § 313, $AG : AC = AC : AE$;

but, § 122, $AG = \frac{1}{2} AB$, and, data, $AE = 2\,AD$;

$\therefore$ substituting, $\frac{1}{2} AB : AC = AC : 2\,AD$;

hence, § 282, $AB : 2\,AC = AC : 2\,AD$, and, § 285, $AB : AC = AC : AD$.

Ex. **91**. $AB$ and $CD$ are two parallel straight lines; $E$ is the middle point of $CD$; $AC$ and $BE$ meet in $F$, and $AE$ and $BD$ meet in $G$. Prove that $FG$ is parallel to $AB$.

**Proof.** § 301, $\triangle$s $ABF$ and $CEF$ are similar,

and, § 299, $BF : EF = AB : CE$; similarly, $AG : EG = AB : ED$,

and since $CE = ED$, $AB : CE = AB : ED$;

hence, $BF : EF = AG : EG$,

and, § 277, $BF - EF : EF = AG - EG : EG$;

that is, $BE : EF = AE : EG$,

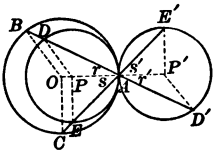
or, § 274, $BE : AE = EF : EG$,

and since, § 59, $\angle AEB = \angle FEG$,

§ 306, $\triangle$s $ABE$ and $GFE$ are similar;

∴ § 299, $\angle ABF = \angle BFG$; hence, § 75, $FG \parallel AB$.

**Ex. 92.** If two circles are tangent to each other, either internally or externally, any two straight lines drawn through the point of contact will be cut proportionally by the circumferences.



**Data:** The Ⓢ whose centers are $O$ and $P$, tangent internally at $A$, and any two lines from $A$, as $AB$ and $AC$, intersected by the circumferences in $B$, $D$, and $C$, $E$ respectively.

**To prove** $AB : AD = AC : AE$.

**Proof.** Draw $OPA$, $OB$, and $PD$.

Since, Ax. 14, $OB = OA$, § 116, $\angle OBA = \angle r$;

similarly, $\angle PDA = \angle r$; hence, $\angle OBA = \angle PDA$.

Then, in $\triangle$s $OBA$ and $PDA$,

$\angle OBA = \angle PDA$, and $\angle r$ is common;

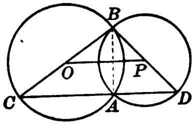
hence, § 301, $\triangle$s $OBA$ and $PDA$ are similar,

and, $AB : AD = OA : PA$; similarly, $AC : AE = OA : PA$;

hence, $AB : AD = AC : AE$.

Similarly, if the Ⓢ are tangent externally, as those whose centers are $O$ and $P'$, it may be shown by similar isosceles $\triangle$s, since, § 59, $\angle r = \angle r'$, and $\angle s = \angle s'$, that any lines through $A$, as $BAD'$ and $CAE'$, are divided proportionally.

**Ex. 93.** From one of the points of intersection of two intersecting circles a diameter of each circle is drawn. Prove (1) that the line joining the extremities of these diameters passes through the other point of intersection; and (2) that this line is parallel to the line of centers of the circles.



**Data:** Ⓢ whose centers are $O$ and $P$ intersecting in $A$ and $B$; their diameters $BC$ and $BD$ respectively; and the line $CD$.

**To prove** 1. $CD$ passes through $A$.
2. $CD \parallel OP$.

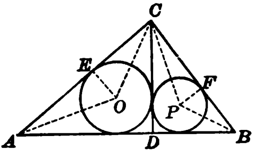
**Proof.** Draw $AB$, $AC$, and $AD$.

Then, § 227, $\angle$s $CAB$ and $BAD$ are rt. $\angle$s;

∴ § 58, $AC$ and $AD$ are in the same straight line; that is, $CD$ passes through $A$.

2. Ax. 14, $CO = OB$, and $DP = PB$; ∴ § 159, $CD \parallel OP$.

Ex. **94.** If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse, and circles are inscribed in the triangles thus formed, the diameters are proportional to the sides of the given right angle.



**Data:** Rt. $\triangle ABC$; $CD \perp AB$ from $C$ the vertex of the rt. $\angle$; and the Ⓢ whose centers are $O$ and $P$ inscribed in $\triangle$s $ADC$ and $BDC$ respectively.

Denote the diameters of these Ⓢ by $D$ and $D'$ respectively.

**To prove** $D : D' = AC : BC$.

**Proof.** Draw $OA$, $OC$, $PB$, and $PC$ and draw the radii $OE$ and $PF$ to the points of contact of $AC$ and $BC$ respectively.

§ 54, $\angle DAC = \angle DCB$; and, Ex. 221, Ax. 7, $\angle OAE = \angle PCF$;

hence, § 302, rt. $\triangle$s $AOE$ and $CPF$ are similar;

hence, $AE : CF = OE : PF$; similarly, $EC : BF = OE : PF$;

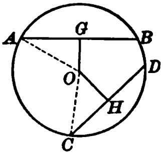
$\therefore$ $AE : CF = EC : BF$, and, § 274, $AE : EC = CF : BF$;

hence, § 276, $AE + EC : EC = CF + BF : BF$, or $AC : EC = BC : BF$;

that is, $AC : BC = EC : BF$; but $EC : BF = OE : PF = 2\,OE : 2\,PF$;

hence, $2\,OE : 2\,PF = AC : BC$; that is, $D : D' = AC : BC$.

Ex. **95.** The distance from the center of a circle to a chord $8^{\text{dm}}$ long is $4^{\text{dm}}$. What is the distance from the center to a chord $5^{\text{dm}}$ long?



**Solution.** Let $AB = 8^{\text{dm}}$, $CD = 5^{\text{dm}}$, and $OG = 4^{\text{dm}}$.

Radius $OA = \sqrt{\overline{AG}^2 + \overline{OG}^2} = \sqrt{4^2 + 4^2} = \sqrt{32}$;

then, $OH = \sqrt{\overline{OC}^2 - \overline{CH}^2} = \sqrt{\overline{OA}^2 - \overline{CH}^2}$

$= \sqrt{(\sqrt{32})^2 - (\frac{5}{2})^2} = \sqrt{32 - \frac{25}{4}} = \frac{1}{2}\sqrt{103}$

$= 5.074+^{\text{dm}}$.

Ex. **96.** If a chord $18^{\text{dm}}$ long is bisected by another chord $22^{\text{dm}}$ long, what are the segments of the latter?

**Solution.** Let $AB = 18^{\text{dm}}$ be bisected at $E$ by chord $CD = 22^{\text{dm}}$.

Then, Ex. 456, $AE : DE = CE : BE$; that is, $9 : DE = CE : 9$;

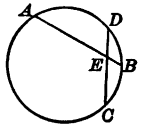
whence, $DE \times CE = 81$; (1) but, data, $DE + CE = 22$; (2)

then, substituting the value of $CE$ from (2) in (1),

$$DE\,(22 - DE) = 81, \text{ or } \overline{DE}^2 - 22\,DE = -81;$$

whence, $DE = 17.324+^{\text{dm}}$, and $CE = CD - DE = 22^{\text{dm}} - 17.324+^{\text{dm}} = 4.676-^{\text{dm}}$.

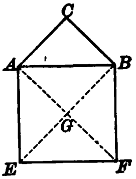
Ex. **97.** If two intersecting chords divide the circumference of a circle into parts whose lengths taken in order are as 1, 1, 2, and 5, what angles do the chords make with each other?



**Solution.** Let chords $AB$ and $CD$ intersect at $E$ and divide the circumference into arcs $CB$, $BD$, $AD$, and $AC$ which are as 1, 1, 2, and 5.

Then, arc $BC = \frac{1}{9}$ of the circum. $= 40°$,
and arc $AD = \frac{2}{9}$ of the circum. $= 80°$;
but, § 230, $\angle CEB$ is measured by $\frac{1}{2}$ (arc $BC +$ arc $AD$);
consequently, $\angle CEB = \frac{1}{2}(40° + 80°) = 60°$;
and, § 55, $\angle DEB = 180° - 60° = 120°$.

Ex. **98.** The square on the hypotenuse of a right isosceles triangle is equivalent to four times the triangle.



**Data:** Isos. rt. $\triangle ABC$ and the square $AEFB$ on its hypotenuse $AB$.

**To prove** $AEFB \approx 4 \triangle ABC$.

**Proof.** Draw the diagonals $AF$ and $BE$ intersecting as at $G$.

Since, § 154, Ex. 110, the diagonals bisect each other at rt. $\angle$s, § 107, they divide the square into four equal rt. isos. $\triangle$s.

Now in $\triangle$s $ABG$ and $ABC$, $AB$ is common, $\angle ABG = \angle ABC$ (§ 111, each being $\frac{1}{2}$ rt. $\angle$); $\therefore$ § 114, $\triangle ABG = \triangle ABC$;
hence, $AEFB \approx 4 \triangle ABG = 4 \triangle ABC$.

Ex. **99.** If the sides of a triangular field are respectively $11^{\text{Hm}}$, $9^{\text{Hm}}$, and $8^{\text{Hm}}$ long, how many hektares are there in the area of the field?

**Solution.** Ex. 632, $A = \sqrt{14 \times 3 \times 5 \times 6} = \sqrt{1260} = 35.496+{}^{\text{Ha}}$.

Ex. **100.** The sides of a triangle are respectively 39, 42, and 45 inches in length. What is the radius of the inscribed circle?

**Solution.** Ex. 632, $A = \sqrt{63 \times 24 \times 21 \times 18} = \sqrt{571536} = 756$ sq. in.

Denoting the radius of the inscribed circle by $r$,
Ex. 562, $A = r\,\frac{1}{2}(39 + 42 + 45) = 63\,r$.
Hence, $63\,r = 756$; whence, $r = \frac{756}{63} = 12$ in.

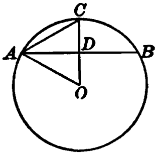
Ex. **101.** The sides of a triangle are respectively 5 ft., 5 ft., and 6 ft. What is the diameter of the circumscribed circle?

**Solution.** Ex. 636, $r = \dfrac{5 \times 5 \times 6}{4\sqrt{8 \times 3 \times 3 \times 2}} = \dfrac{5 \times 5 \times 6}{4 \times 12} = 3\frac{1}{8}$ ft.
hence, the diameter $= 2\,r = 6\frac{1}{4}$ ft.

Ex. **102.** A triangular field has its sides respectively 16 rd., 24 rd., and 36 rd. long. What is the length of a line from the middle of the longest side to the opposite corner? What is the area of the field?

**Solution.** (1) Ex. 633, $m = \frac{1}{2}\sqrt{2\,(16^2 + 24^2) - 36^2} = \frac{1}{2}\sqrt{368} = 9.59+$ rd.
(2) Ex. 632, $A = \sqrt{38 \times 22 \times 14 \times 2} = \sqrt{23408} = 153$ sq. rd., nearly.

**Ex. 103.** If a chord $10^{cm}$ long is $5^{cm}$ distant from the center of a circle, what is the radius of the circle and the distance from the end of the chord to the end of the radius that is perpendicular to the chord?



**Solution.** Let chord $AB = 10^{cm}$, let radius $OC$ be $\perp$ $AB$ at $D$, and let $OD = 5^{cm}$. Draw $AC$ and the radius $OA$.

§ 349, $OA = \sqrt{\overline{AD}^2 + \overline{OD}^2} = \sqrt{5^2 + 5^2} = \sqrt{50}$;

then, $OC = \sqrt{50}$, and $CD = OC - OD = \sqrt{50} - 5$.

Then, $AC = \sqrt{\overline{AD}^2 + \overline{CD}^2} = \sqrt{5^2 + (\sqrt{50} - 5)^2} = \sqrt{100 - 50\sqrt{2}}$
$= \sqrt{29.28935} = 5.41+^{cm}$.

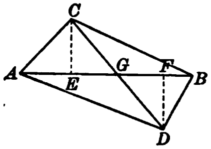
**Ex. 104.** How many square meters are there in the area of the quadrilateral $ABCD$, if $AB = 6^m$, $BC = 11^m$, $CD = 4^m$, $AD = 13^m$, and the diagonal $AC = 15^m$?

**Solution.** Ex. 632, area $\triangle ABC = \sqrt{16 \times 10 \times 5 \times 1} = \sqrt{800} = 28.284+^{sq\,m}$,

and area $\triangle ACD = \sqrt{16 \times 1 \times 12 \times 3} = \sqrt{576} = 24^{sq\,m}$.

Hence, area $ABCD = 24^{sq\,m} + 28.284+^{sq\,m} = 52.284+^{sq\,m}$.

**Ex. 105.** If two equivalent triangles have a common base, and lie on opposite sides of it, the base, or the base produced, will bisect the line joining their vertices.



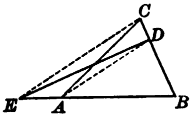
**Data:** Equivalent $\triangle$s $ABC$ and $ABD$ on the common base $AB$, and the line $CD$ cut by $AB$ at $G$.

**To prove** $CG = DG$.

**Proof.** Draw $CE$ and $DF \perp AB$.

§ 335, area $\triangle ABC = \frac{1}{2} AB \times CE$, and area $\triangle ABD = \frac{1}{2} AB \times DF$;
hence, data, $\frac{1}{2} AB \times CE = \frac{1}{2} AB \times DF$; whence, $CE = DF$,
and since, § 59, $\angle EGC = \angle FGD$, § 114, $\triangle EGC = \triangle FGD$, and $CG = DG$.

**Ex. 106.** $ABC$ is a given triangle. Construct an equivalent triangle, having given its vertex at a given point in $BC$, and its base in the same straight line as $AB$.



**Solution.** Let $D$ be the given point in $BC$.

Draw $DA$, and draw $CE \parallel DA$, meeting $BA$ produced as in $E$. Draw $ED$.

Then, $\triangle EBD$ is the $\triangle$ required.

**Proof.** Ex. 481, $\triangle EAD \Leftrightarrow \triangle ADC$;
hence, $\triangle ABD + \triangle EAD \Leftrightarrow \triangle ABD + \triangle ADC$;
that is, $\triangle EBD \Leftrightarrow \triangle ABC$.

**Ex. 107.** Through the vertex $A$ of the parallelogram $ABCD$ draw a line meeting the side $CB$ produced in $F$, and the side $CD$ produced in $E$. Prove that the rectangle of the produced parts of the sides is equivalent to the rectangle of the sides.

**Proof.** Since $AD \parallel FC$, § 289, $ED : DC = EA : AF$,
and since $AB \parallel EC$, $BC : BF = EA : AF$;
hence, $ED : DC = BC : BF$; and, §§ 269, 324, rect. $ED \cdot BF \Leftrightarrow$ rect. $DC \cdot BC$.

Ex. **108**. $ABC$ is a right triangle having its right angle at $B$. At $A$ and $C$ perpendiculars to $AC$ are erected to meet $CB$ and $AB$ produced in $E$ and $F$ respectively, and $EF$ is drawn. Prove that the triangles $BEF$ and $ABC$ are equivalent.

**Proof.** Since, § 71, $AE \parallel CF$, $AEFC$ is a trapezoid, and its diagonals $AF$ and $CE$ intersect at $B$; hence, Ex. 559, $\triangle BEF \Leftrightarrow \triangle ABC$.

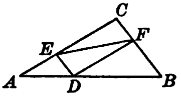
Ex. **109**. The square upon the altitude of an equilateral triangle is equivalent to three times the square upon half of one of the sides of the triangle.

**Data**: Equilateral $\triangle ABC$, whose altitude is $CD$.

**To prove** $\overline{CD}^2 \Leftrightarrow$ three times the square upon $\frac{1}{2} AB$.

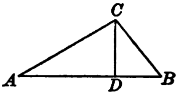
**Proof.** § 349, $\overline{CD}^2 \Leftrightarrow \overline{AC}^2 - \overline{AD}^2$;
but, § 122, $AC = AB = 2\,AD$; $\therefore \overline{CD}^2 \Leftrightarrow 4\,\overline{AD}^2 - \overline{AD}^2 \Leftrightarrow 3\,\overline{AD}^2$.

Ex. **110**. If from a point $D$ in the base $AB$ of the triangle $ABC$ straight lines are drawn parallel to the sides $AC$ and $BC$ respectively, so as to meet $BC$ in $F$ and $AC$ in $E$, triangle $EFC$ is a mean proportional between triangles $ADE$ and $DBF$.



**Proof.** § 76, $\angle AED = \angle ECF$;
$\therefore$ § 340, $\triangle ADE : \triangle EFC = AE \times DE : EC \times FC$,
and since, § 153, $DE = FC$, $\triangle ADE : \triangle EFC = AE : EC$; (1)
similarly, $\triangle EFC : \triangle DBF = FC : BF$; (2)
but since, § 301, $\triangle ADE$ and $DBF$ are similar,
§ 299, $AE : DF = DE : BF$; that is, $AE : EC = FC : BF$;
hence, from (1) and (2), $\triangle ADE : \triangle EFC = \triangle EFC : \triangle DBF$.

Ex. **111**. Two sides of a triangle are $70^{m}$ and $65^{m}$, and the difference of the segments of the other side made by a perpendicular from the opposite vertex is $9^{m}$. What is the length of the other side?



**Solution.** Let $AC = 70^{m}$, $BC = 65^{m}$,
and $AD - BD = 9^{m}$.

Then, § 349, $\overline{AD}^2 = \overline{AC}^2 - \overline{DC}^2$; that is, $\overline{AD}^2 = 70^2 - \overline{DC}^2$; (1)
also $\overline{BD}^2 = \overline{BC}^2 - \overline{DC}^2$; that is, $\overline{BD}^2 = 65^2 - \overline{DC}^2$; (2)
$\therefore$ subtracting, $\overline{AD}^2 - \overline{BD}^2 = 70^2 - 65^2 = 675$. (3)
Data, $AD - BD = 9$. (4)

Dividing (3) by (4), $AD + BD = 75$; that is, $AB = 75^{m}$.

**Ex. 112.** The sum of two sides of a triangle is 128 ft., and a perpendicular from the vertex opposite the other side divides that side into segments of 60 ft. and 28 ft. What are the sides of the triangle?

**Solution.** (See Fig. for Ex. 111, Rev.) Let $AC + BC = 128$ ft.; $AD = 60$ ft.; and $BD = 28$ ft.

Then, § 349, $\overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2$; that is, $\overline{AC}^2 = 60^2 + \overline{CD}^2$; (1)

also, $\overline{BC}^2 = \overline{BD}^2 + \overline{CD}^2$; that is, $\overline{BC}^2 = 28^2 + \overline{CD}^2$; (2)

∴ subtracting, $\overline{AC}^2 - \overline{BC}^2 = 60^2 - 28^2 = 2816.$ (3)

Data, $AC + BC = 128.$ (4)

Dividing (3) by (4), $AC - BC = 22.$ (5)

Taking half the sum of (4) and (5), $AC = 75$ ft.; taking half the difference, $BC = 53$ ft.

**Ex. 113.** Two sides of a triangle are in proportion to each other as 6 is to 5, and the adjacent segments of the other side made by a perpendicular from the opposite vertex are 36 ft. and 14 ft., respectively. What are the sides?

**Solution.** (See Fig. for Ex. 111, Rev.) Let $AD = 36$ ft., $BD = 14$ ft., and $AC : BC = 6 : 5$.

Then, § 349, $\overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2$; that is, $\overline{AC}^2 = 36^2 + \overline{CD}^2$; (1)

also, $\overline{BC}^2 = \overline{BD}^2 + \overline{CD}^2$; that is, $\overline{BC}^2 = 14^2 + \overline{CD}^2$; (2)

hence, $\overline{AC}^2 : \overline{BC}^2 = 36^2 + \overline{CD}^2 : 14^2 + \overline{CD}^2$;

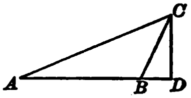
but, data, $AC : BC = 6 : 5$; hence, § 288, $\overline{AC}^2 : \overline{BC}^2 = 36 : 25$;

∴ $36^2 + \overline{CD}^2 : 14^2 + \overline{CD}^2 = 36 : 25$; hence, $36(14^2 + \overline{CD}^2) = 25(36^2 + \overline{CD}^2)$;

that is, $7056 + 36\,\overline{CD}^2 = 32400 + 25\,\overline{CD}^2$; whence, $\overline{CD}^2 = 2304$.

Substituting in (1), $\overline{AC}^2 = 1296 + 2304 = 3600$; whence, $AC = \sqrt{3600} = 60$ ft.; substituting in (2), $\overline{BC}^2 = 196 + 2304 = 2500$; whence, $BC = \sqrt{2500} = 50$ ft.

**Ex. 114.** The difference of the two sides of an oblique triangle, obtuse-angled at the base, is $9^{\mathrm{m}}$, and the segments of the base produced made by a perpendicular from the opposite vertex are $30^{\mathrm{m}}$ and $9^{\mathrm{m}}$. What are the sides?



**Solution.** Let $\triangle ABC$ be an obtuse $\triangle$ in which $\angle B$ is the obtuse $\angle$ and $AC - BC = 9^{\mathrm{m}}$; also let the $\perp CD$ meet $AB$ produced so that $AD = 30^{\mathrm{m}}$ and $BD = 9^{\mathrm{m}}$.

Then, § 349, $\overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2$; that is, $\overline{AC}^2 = 30^2 + \overline{CD}^2$;

also, $\overline{BC}^2 = \overline{BD}^2 + \overline{CD}^2$; that is, $\overline{BC}^2 = 9^2 + \overline{CD}^2$;

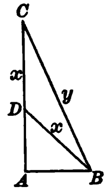
subtracting, $\overline{AC}^2 - \overline{BC}^2 = 30^2 - 9^2 = 819.$ (1)

Data, $AC - BC = 9.$ (2)

Dividing (1) by (2), $AC + BC = 91.$ (3)

Whence, as in Ex. 112, Rev., $AC = 50^{\mathrm{m}}$, and $BC = 41^{\mathrm{m}}$.

**Ex. 115.** A flag pole 140 ft. long, standing on an eminence 30 ft. high, broke so that the top struck the level ground at a distance from the base of the pole equal to the length of the part standing. What was the length of the part broken off?



**Solution.** In rt. $\triangle ABC$ let $AD = 30$ ft. represent the eminence, $DC$ the part of the pole left standing, and $BC$ the part broken off.

Denote $DC$ by $x$, and $BC$ by $y$; then, $BD = x$, and $x + y = 140$ ft.

§ 349, $\overline{BC}^2 - \overline{AC}^2 = \overline{AB}^2$; that is, $y^2 - (30 + x)^2 = \overline{AB}^2$;

also, $\overline{BD}^2 - \overline{AD}^2 = \overline{AB}^2$; that is, $x^2 - 30^2 = \overline{AB}^2$;

hence, $x^2 - 30^2 = y^2 - (30 + x)^2$. (1)

Data, $x + y = 140$; whence, $y = 140 - x$, and $y^2 = (140 - x)^2$.

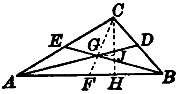
Substituting in (1), $x^2 - 30^2 = (140 - x)^2 - (30 + x)^2$;

that is, $x^2 - 900 = 19600 - 280x + x^2 - 900 - 60x - x^2$;

transposing, etc., $x^2 + 340x = 19600$; whence, $x = 50.227+$ ft.;

consequently, $y = 140$ ft. $- 50.227+$ ft. $= 89.773-$ ft.

**Ex. 116.** If from the extremities of the base of any triangle lines are drawn bisecting the other two sides, these lines intersect within the triangle and form another triangle on the same base equivalent to one third of the original triangle.



**Data:** $\triangle ABC$, and the medians $AD$ and $BE$ drawn from the extremities of the base, and intersecting each other at $G$.

**To prove** $\triangle ABG \Leftrightarrow \frac{1}{3} \triangle ABC$.

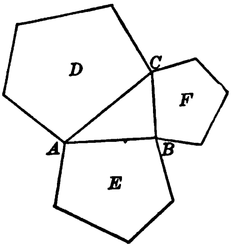
**Proof.** Draw the median $CGF$ and the altitude $CH$, and from $G$ draw $GJ \parallel AB$, meeting $CH$ as in $J$.

Then, §§ 290, 168, $CH : JH = CF : GF = 3 : 1$;

but, § 336, $\triangle ABC : \triangle ABG = CH : JH$;

that is, $\triangle ABC : \triangle ABG = 3 : 1$; whence, $\triangle ABG \Leftrightarrow \frac{1}{3} \triangle ABC$.

**Ex. 117.** Upon the sides of a right triangle, as homologous sides, three similar polygons of any number of sides are constructed. Prove that the polygon upon the hypotenuse is equivalent to the sum of the polygons upon the other two sides.



**Data:** Rt. $\triangle ABC$, and the similar polygons $D$, $E$, and $F$, of any number of sides, constructed upon $AC$, $AB$, and $BC$ respectively, as homologous sides.

**To prove** $D \Leftrightarrow E + F$.

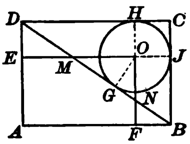
**Proof.** § 344, $E : D = \overline{AB}^2 : \overline{AC}^2$,

and $F : D = \overline{BC}^2 : \overline{AC}^2$;

hence, § 280, $E + F : D = \overline{AB}^2 + \overline{BC}^2 : \overline{AC}^2$;

§ 349, $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$; $\therefore$ § 271, $E + F = D$; that is, $E + F \Leftrightarrow D$.

Ex. **118**. *ABCD* is a rectangle, and *BD* is its diagonal; a circle whose center is *O* is inscribed in the triangle *DBC*; *EO* and *FO* are drawn perpendicular to *AD* and *AB* respectively. Then, the rectangle *AFOE* is equivalent to one half the rectangle *ABCD*.



**Proof.** Draw the radii *OG*, *OJ*, and *OH* to the points of contact.

Then, in rt. $\triangle$s *MGO* and *EMD*,

$OG = OH = ED$, and, § 59, $\angle GMO = \angle EMD$

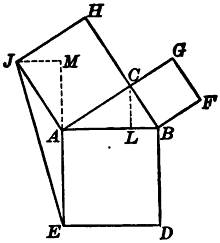
hence, § 114, $\triangle MGO = \triangle EMD$.

Similarly, $\triangle GNO = \triangle FBN$.

Hence, $AFNME + \triangle MGO + \triangle GNO \Bumpeq AFNME + \triangle EMD + \triangle FBN$;

that is, § 152, $AFOE \Bumpeq \triangle ABD \Bumpeq \frac{1}{2} ABCD$.

Ex. **119**. If squares are described upon the three sides of a right triangle, and the extremities of the adjacent sides of any two squares are joined, the triangle so formed is equivalent to the given triangle.



**Data:** Rt. $\triangle ABC$; the squares *ABDE*, *BCGF*, and *ACHJ* on its sides; and the line *JE* joining the extremities of the adjacent sides of two of the squares.

**To prove** $\triangle AEJ \Bumpeq \triangle ABC$.

**Proof.** Draw $JM \perp EA$ produced, and draw $CL \perp AB$.

Then, $\angle JAM$ is comp. of $\angle MAC$,

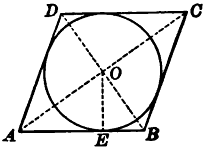
and $\angle LAC$ is comp. of $\angle MAC$;

$\therefore$ § 54, $\angle JAM = \angle LAC$, and since $AJ = AC$, § 114, $\triangle AMJ = \triangle ALC$,

and $JM = CL$; since $AE = AB$, § 336, $\triangle AEJ \Bumpeq \triangle ABC$.

Ex. **120**. Inscribe a circle in a given rhombus.

**Solution.** Draw the diagonals *AC* and *BD*, of the given rhombus *ABCD*, intersecting as at *O*.

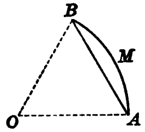


Draw $OE \perp AB$, and with *O* as center and radius *OE* describe a ⊙.

Then, this ⊙ is inscribed in *ABCD*.

**Proof.** Since, Ex. 154, *AC* bisects the $\angle$s *A* and *C*, and *BD* bisects the $\angle$s *B* and *D*, § 134, every point in *AC* is equidistant from *AB* and *AD*, also from *CB* and *CD*; and every point in *BD* is equidistant from *AB* and *BC*, also from *AD* and *DC*; hence, *O*, the intersection of *AC* and *BD*, is equidistant from *AB*, *BC*, *DC*, and *AD*; consequently, the ⊙ whose center is *O* and radius $= OE$, is tangent to the sides, and hence inscribed in *ABCD*.

**Ex. 121.** A segment whose arc is 60° is cut off from a circle whose radius is 15 ft. What is the area of the segment?



**Solution.** Since $\angle O = 60°$, sect. $AOBM = \frac{1}{6}$ of the $\odot$ $= \frac{1}{6}$ of $\pi R^2 = \frac{1}{6} \times 3.1416 \times 15^2 = 117.81$ sq. ft.

Since, Ax. 14, $OA = OB$,

§ 116, $\angle OAB = \angle OBA = \frac{1}{2}(180° - 60°) = 60°$;

consequently, § 119, $\triangle AOB$ is an equilateral $\triangle$ whose side is 15 ft., and, Ex. 632, its area $= \sqrt{\frac{45}{2} \times \frac{15}{2} \times \frac{15}{2} \times \frac{15}{2}} = \frac{225}{4}\sqrt{3} = 97.4278+$ sq. ft.

Then, area of seg. $ABM$ = area of sect. $AOBM$ − area of $\triangle AOB$
$= 117.81$ sq. ft. $- 97.4278+$ sq. ft. $= 20.38+$ sq. ft.

**Ex. 122.** If the bisectors of all the angles of a polygon meet in a point, a circle may be inscribed in the polygon.

**Proof.** § 134, every point in each of the angle bisectors is equidistant from the sides of the respective $\angle s$; $\therefore$ the intersection of the bisectors is equidistant from the sides of the polygon.

Hence, a circle described with the intersection as a center and a radius equal to a perpendicular from that point to any side will be inscribed in the polygon, for each of the sides of the polygon will be perpendicular to a radius at its extremity, and, § 205, tangent to the circle.

**Ex. 123.** If the area of a certain circle is $154^{\text{sq m}}$, how many degrees are there in an angle at the center, if it is subtended by an arc of the circumference $5.5^{\text{m}}$ long?

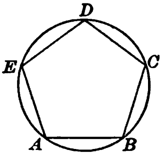
**Solution.** § 398, area $= \pi R^2 = 154$; whence, $R = \sqrt{\frac{154}{\pi}}$;

hence, § 395, $C = 2\pi R = 2\pi\sqrt{\frac{154}{\pi}} = 2\sqrt{154\pi} = 2\sqrt{483.8064} = 43.99+^{\text{m}}$;

consequently,

a central $\angle$ subtended by an arc $5.5^{\text{m}}$ long equals $\frac{5.5}{43.99}$ of 360°, or 45°, nearly.

**Ex. 124.** Prove that an equiangular polygon inscribed in a circle is regular, if the number of its sides is odd.



**Data:** Equiangular polygon of an odd number of sides, as $ABCDE$, inscribed in a circle.

**To prove** $ABCDE$ regular.

**Proof.** § 225, $\angle ABC$ is measured by $\frac{1}{2}$ arc $AEDC$,

and $\angle BCD$ is measured by $\frac{1}{2}$ arc $BAED$;

but, data, $\angle ABC = \angle BCD$; hence, arc $AEDC$ = arc $BAED$;

$\therefore$ taking away arc $AED$ which is common to both,

arc $DC$ = arc $AB$; $\therefore$ § 196, chord $DC$ = chord $AB$.

Similarly, any two alternate sides are equal, and since the number of sides is odd, $ABCDE$ is equilateral, and, § 374, $ABCDE$ is regular.

**Ex. 125.** Two parallel chords in a circle are the sides of regular inscribed polygons, one a hexagon and the other a dodecagon. If the radius of the circle is 11 in., how far apart are the chords?

**Solution.** Ex. 648, apothem of hexagon $= \frac{1}{2} r\sqrt{3}$,

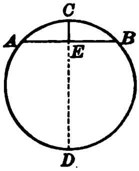
and, Ex. 712, apothem of dodecagon $= \frac{r}{2}\sqrt{2+\sqrt{3}}$;

hence,

$$\text{the distance between the chords} = \frac{r}{2}\sqrt{2+\sqrt{3}} - \tfrac{1}{2} r\sqrt{3}$$
$$= \tfrac{11}{2}\sqrt{2 + 1.732+} - \tfrac{11}{2}(1.732+)$$
$$= \tfrac{11}{2}(1.9318+) - \tfrac{11}{2}(1.732+) = 1.0989+ \text{ in.}$$

If chords are on opposite sides of the center the distance is 20.1509+ in.

**Ex. 126.** What is the length of the side of a square equivalent to a circle in which a chord of $30^{\text{dm}}$ has an arc whose height is $5^{\text{dm}}$?



**Solution.** Let chord $AB = 30^{\text{dm}}$, and let $CD$ be the diameter $\perp AB$ intersecting it at $E$; then, $CE = 5^{\text{dm}}$.

§ 314,

$DE : AE = AE : CE$; that is, § 200, $DE : 15 = 15 : 5$;

whence,

$DE = \dfrac{15 \times 15}{5} = 45^{\text{dm}}$, and the diam. $CD = DE + CE = 45^{\text{dm}} + 5^{\text{dm}} = 50^{\text{dm}}$;

consequently, the radius $R = \frac{1}{2}$ of $50^{\text{dm}}$, or $25^{\text{dm}}$;

hence, § 398, the area of the $\odot = \pi R^2 = 3.1416 \times 25^2 = 1963.5^{\text{sq dm}}$;

but, data, the area of the square $= 1963.5^{\text{sq dm}}$;

consequently, its side $= \sqrt{1963.5} = 44.31+^{\text{dm}}$.

**Ex. 127.** If a 4-inch pipe will fill a cistern in 2 hr. 30 min., how long will it take a 2-inch pipe to fill it?

**Solution.** Since the volume of water flowing from the two pipes is in proportion to the areas of their openings, and since, § 399, these areas are to each other as $4^2 : 2^2$, denoting the number of hours required by the 2-inch pipe by $x$,

$$x : 2\tfrac{1}{2} = 4^2 : 2^2; \text{ whence, } x = \frac{2\frac{1}{2} \times 4^2}{2^2} = 10 \text{ hr.}$$

**Ex. 128.** If an equilateral triangle and a regular decagon each has a perimeter of $6^{\text{m}}$, what is the difference in area between them?

**Solution.** Ex. 649, side of $\Delta = 6 \div 3 = 2 = r\sqrt{3}$; whence, $r = \dfrac{2}{\sqrt{3}}$;

then, apothem $= \frac{1}{2} r = \dfrac{1}{\sqrt{3}}$, and, § 389, area $= \frac{1}{2} \times 6 \times \dfrac{1}{\sqrt{3}} = \sqrt{3} = 1.732+^{\text{sq m}}$.

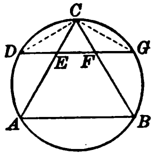
Ex. 711, side of decagon$=6 \div 10 = \frac{3}{5} = \frac{r}{2}(\sqrt{5}-1)$; whence, $r = \frac{6}{5(\sqrt{5}-1)}$;

$$\text{then, apothem} = \frac{r}{4}\sqrt{10+2\sqrt{5}} = \frac{\frac{6}{5(\sqrt{5}-1)}}{4}\sqrt{10+2\sqrt{5}} =$$

$$\frac{1}{4.12} \times 3.8 = \frac{1.9}{2.06} \text{ and area} = \tfrac{1}{2} \times 6 \times \frac{1.9}{2.06} = 2.767+^{\text{sq m}}.$$

Hence, difference of area $= 2.767^{\text{sq m}} - 1.732^{\text{sq m}} = 1.035+^{\text{sq m}}$.

Ex. **129**. If an equilateral triangle is inscribed in a circle, the line joining the middle points of the arcs cut off by two of its sides will be trisected by those sides.



**Data**: Inscribed equilateral $\triangle ABC$; $D$ and $G$ the middle points of arcs $AC$ and $BC$ respectively; and the line $DG$ cut by $AC$ in $E$ and by $BC$ in $F$.

**To prove** $DE = EF = FG$.

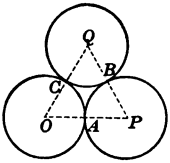
**Proof**. Draw $DC$ and $GC$. Since arc $CG$ = arc $AD$, § 225, $\angle D = \angle DCE$, and, § 118, $DE = CE$;

similarly, $FG = CF$.

§§ 230, 225, $\angle CEF = \angle CFE = \angle ECF$ (each being measured by $\frac{1}{6}$ of a circumference);

hence, § 119, $CE = CF = EF$; consequently, Ax. 1, $DE = EF = FG$.

Ex. **130**. If the area between three equal circles, each tangent to the other two, is $40^{\text{sq m}}$, what are the radii of the circles?



**Solution**. As in Ex. 722,

area of $ABC$ = area $\triangle OPQ - 3$ area sect. $OAC$,

and each side of $\triangle OPQ$ is $2R$;

consequently, Ex. 632,

area $\triangle OPQ = \sqrt{3R \times R \times R \times R} = R^2\sqrt{3}$.

Since $\angle O = 60°$, area sect. $OAC = \frac{1}{6}$ area of $\odot = \frac{1}{6}\pi R^2$.

Hence, area $ABC = R^2\sqrt{3} - 3 \times \frac{1}{6}\pi R^2 = R^2(\sqrt{3} - \frac{1}{2}\pi)$;

but, data, area $ABC = 40^{\text{sq m}}$; then, $R^2(\sqrt{3} - \frac{1}{2}\pi) = 40$;

whence, $R^2 = \frac{40}{\sqrt{3} - \frac{1}{2}\pi} = 248.1389+$, and $R = \sqrt{248.1389} = 15.75+^{\text{m}}$.

Ex. **131**. Construct a circle equal to three fourths of a given circle.

**Solution**. Denote the given $\odot$ by $A$ and its radius by $R$.

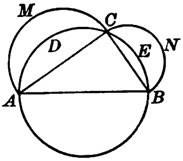
As in Ex. 737, construct $\sqrt{3}$ when $R$ is the unit, and with $R' = \frac{1}{2}$ of this line construct $\odot B$.

Then, $B$ is the $\odot$ required.

**Proof**. § 398, $A = \pi R^2$, and $B = \pi R'^2 = \pi(\frac{1}{2}R\sqrt{3})^2 = \frac{3}{4}\pi R^2$;

hence, $B \Leftrightarrow \frac{3}{4}A$.

Ex. **132**. If a circle is circumscribed about a right triangle, and on each of its sides as a diameter a semicircle is described exterior to the triangle, the sum of the areas of the crescents thus formed is equal to the area of the triangle.

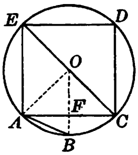


**Data:** Rt. $\triangle ABC$ and the semicircles on its sides as diameters, forming the crescents $AMCD$ and $BNCE$.

**To prove** area $AMCD$ + area $BNCE$ = area $\triangle ABC$.

**Proof.** Semicircle on $AC$ + semicircle on $BC$ = $\frac{1}{8}\pi\overline{AC}^2 + \frac{1}{8}\pi\overline{BC}^2 = \frac{1}{8}\pi(\overline{AC}^2 + \overline{BC}^2) = \frac{1}{8}\pi\overline{AB}^2$; but semicircle on $AB = \frac{1}{8}\pi\overline{AB}^2$; $\therefore$ semicircle on $AC$ + semicircle on $BC$ = semicircle on $AB$; hence, subtracting segments $ACD$ and $BCE$ from each side of this equation, area $AMCD$ + area $BNCE$ = area $\triangle ABC$.

Ex. **133**. The area of an inscribed regular octagon is equal to that of a rectangle whose sides are respectively equal to the sides of the inscribed and circumscribed squares.



**Data:** $AB$ the side of an inscribed regular octagon; $AC$ the side of the inscribed square $ACDE$; and $EC$ its diagonal which is the diameter of the ⊙ and equal to the side of the circumscribed square.

**To prove** area of octagon $= AC \times EC$.

**Proof.** Draw $OA$; also draw $OB$ intersecting $AC$ as at $F$.

As in Ex. 649, $OB \perp AC$ at its middle point;

then, $AF = \frac{1}{2}AC$, and, Ax. 14, $OB = \frac{1}{2}EC$.

Then, § 335, area $\triangle ABO = \frac{1}{2}OB \times AF = \frac{1}{2} \times \frac{1}{2}EC \times \frac{1}{2}AC = \frac{1}{8}AC \times EC$;

but area $\triangle ABO = \frac{1}{8}$ area of the octagon;

hence, area of the octagon $= AC \times EC$.

Ex. **134**. If the radius of a circle is $r$, prove that the area of a regular inscribed octagon is $2r^2\sqrt{2}$.

**Proof.** Ex. 710, the side $= r\sqrt{2-\sqrt{2}}$;

consequently, the perimeter $= 8r\sqrt{2-\sqrt{2}}$;

also, the apothem $= \frac{r}{2}\sqrt{2+\sqrt{2}}$;

hence, § 389, area of the octagon $= \frac{1}{2} \times 8r\sqrt{2-\sqrt{2}} \times \frac{r}{2}\sqrt{2+\sqrt{2}}$

$$= 2r^2\sqrt{2-\sqrt{2}} \times \sqrt{2+\sqrt{2}}$$

$$= 2r^2\sqrt{(2-\sqrt{2})(2+\sqrt{2})} = 2r^2\sqrt{2}.$$

Ex. **135**. The area of a circle is a mean proportional between the areas of any two similar polygons, one of which is circumscribed about the circle and the other is isoperimetric with the circle.

**Proof.** Denote the polygons by $A$ and $B$, their perimeters by $P$ and $P'$, and the radii of their inscribed circles by $R$ and $R'$, respectively.

Then, § 345, $A : B = P^2 : P'^2$, and $A : B = R^2 : R'^2$;

hence, $P^2 : P'^2 = R^2 : R'^2$, or, § 288, $P : P' = R : R'$;

and since, data, $P' = C$, $P : C = R : R'$;

$\therefore$ multiplying the terms of the first couplet by $\frac{1}{2} R$, and of the second by $\frac{1}{2} C$,

§ 282, $\frac{1}{2} PR : \frac{1}{2} CR = \frac{1}{2} CR : \frac{1}{2} CR'$;

that is, the area of the circle is a mean proportional between the areas of the polygons.

Ex. **136**. Prove that the sum of the perpendiculars drawn to the sides of a regular polygon from any point within is equal to the apothem of the polygon multiplied by the number of its sides.

**Data**: A regular polygon of $n$ sides, each side equal to $AB$; and the $\perp$s from $P$, any point within, to the sides.

Denote the apothem by $a$ and the $\perp$s by $h$, $h'$, $h''$, etc.

**To prove** $h + h' + h'' + \text{etc.} = a \times n$.

**Proof.** Draw lines from $P$ to the vertices, dividing the polygon into $\triangle$s whose altitudes are $h$, $h'$, $h''$, etc., and bases the sides of the polygon, each being equal to $AB$.

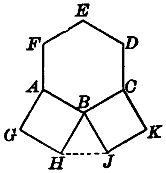
Then, area of polygon = sum of areas of $\triangle$s $= \frac{1}{2} AB(h + h' + h'' + \text{etc.})$;

but, § 389, area of polygon $= \frac{1}{2} n \times AB \times a$;

hence, $\frac{1}{2} AB(h + h' + h'' + \text{etc.}) = \frac{1}{2} n \times AB \times a$;

whence, $h + h' + h'' + \text{etc.} = a \times n$.

Ex. **137**. If upon the sides of a regular hexagon squares are constructed outwardly, the exterior vertices of these squares are the vertices of a regular dodecagon.



**Data**: Regular hexagon $ABCDEF$ and the squares $AH$, $BK$, etc., on its sides.

**To prove** $G$, $H$, $J$, $K$, etc., the vertices of a regular dodecagon.

**Proof.** Draw $HJ$.

§ 167, $\angle ABC =$ sup. of $\frac{1}{6}$ of 4 rt. $\angle$s = sup. of $60° = 120°$, $\angle ABH = 90°$, and $\angle JBC = 90°$; $\therefore$ $\angle HBJ = 360° - (120° + 90° + 90°) = 60°$.

Now, $HB = JB$ (being sides of equal squares);

$\therefore$ § 116, $\angle BHJ = \angle BJH = \frac{1}{2}(180° - 60°) = 60°$;

hence, § 119, $BH = HJ$, and $GH = HJ = JK =$ etc.;

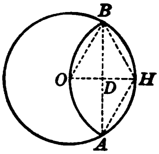
that is, the polygon is equilateral.

Ax. 2, $\angle GHB + \angle BHJ = \angle BJK + \angle BJH =$ etc.,

or $\angle GHJ = \angle HJK =$ etc.;

hence, § 374, $G$, $H$, $J$, $K$, etc., are the vertices of a regular polygon, which is a dodecagon.

Ex. **138**. A horse is tethered to a hook on the inner side of a fence which bounds a circular grass plot. His tether is so long that he can just reach the center of the plot. The area of so much of the plot as he can graze over is $\frac{98}{3}(4\pi - 3\sqrt{3})$ sq. rd.; find the length of the tether and the circumference of the plot.



**Solution.** (1) $\triangle OHB$ is equilateral, each side being $R$.

Then, Ex. 74, $\angle OHB = 60°$; similarly, $\angle OHA = 60°$;

$\therefore$ $\angle AHB = 120°$, and $HAOB$ is a sector whose $\angle$ is 120°;

consequently, its area $= \frac{120}{360} \times \pi R^2 = \frac{1}{3}\pi R^2$.

§ 106, $OH$ and $AB$ bisect each other and are $\perp$ to each other at $D$;

then, § 350, $BD = \sqrt{\overline{BH}^2 - \overline{DH}^2} = \sqrt{R^2 - (\frac{1}{2}R)^2} = R\sqrt{3}$, and $AB = 2R\sqrt{3}$;

consequently, area $\triangle ABH = \frac{1}{2} \times 2R\sqrt{3} \times \frac{1}{2}R = \frac{1}{2}R^2\sqrt{3}$;

hence, area of segment $ABO = \frac{1}{3}\pi R^2 - \frac{1}{2}R^2\sqrt{3}$,

and the area grazed over $= 2(\frac{1}{3}\pi R^2 - \frac{1}{2}R^2\sqrt{3})$.

Hence, data, $2(\frac{1}{3}\pi R^2 - \frac{1}{2}R^2\sqrt{3}) = \frac{98}{3}(4\pi - 3\sqrt{3})$;

$\therefore$ $\frac{1}{3}(2\pi - 3\sqrt{3})R^2 = \frac{98}{3}(4\pi - 3\sqrt{3})$;

whence, $$R^2 = \frac{98(4\pi - 3\sqrt{3})}{2\pi - 3\sqrt{3}} = 664.4445+,$$

and $R = \sqrt{664.4445+} = 25.77+$ rd., the length of the tether.

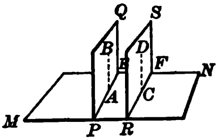
(2) Circum. of plot $= 2\pi R = 2 \times 3.1416 \times 25.77+ = 161.918+$ sq. rd.

Ex. **139**. If equal straight lines are drawn from a given point to a given plane, they make equal angles with the plane.

**Proof.** Pass a plane through any two of the equal lines, and its intersection with the plane is the base of an isos. $\triangle$ whose equal sides are the two lines.

Then, Ex. 766, these lines make equal angles with the plane.

Ex. **140**. Two planes which are each perpendicular to a third plane are parallel, if their intersections with the third plane are parallel.



**Data**: Planes $PQ$ and $RS \perp MN$ and intersecting $MN$ in the parallel lines $PE$ and $RF$.

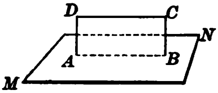
**To prove** $PQ \parallel RS$.

**Proof.** Draw $AB$ and $CD \perp MN$ at points in $PE$ and $RF$ respectively.

Then, § 481, $AB$ lies in $PQ$ and $CD$ in $RS$,

and, § 453, $AB \parallel CD$; but, data, $PE \parallel RF$; hence, § 469, $PQ \parallel RS$.

**Ex. 141.** The line joining the extremities of two equal lines which are perpendicular to a plane, on the same side of it, is parallel to the plane.



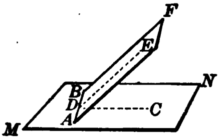
**Data:** Equal lines $AD$ and $BC \perp$ plane $MN$, and the line $DC$ joining their extremities.

**To prove** $DC \parallel MN$.

**Proof.** § 453, $AD \parallel BC$; hence, § 63, $AD$ and $BC$ are in the same plane, § 427, $DC$ lies in that plane, and $AB$ is the intersection of plane $AC$ with $MN$.

Since, $AD$ and $BC$ are equal and parallel, § 150, $ABCD$ is a parallelogram, and $DC \parallel AB$; hence, § 457, $DC \parallel MN$.

**Ex. 142.** Through a given line in a given plane pass a plane to make a given angle with the given plane.



**Solution.** In the given plane $MN$ draw any line, as $CD \perp AB$, the given line; draw $DE \perp AB$ and making $\angle CDE =$ the given $\angle$.

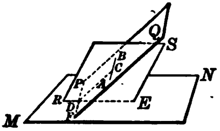
Pass the plane $AF$ through $AB$ and $DE$.

Then, $AF$ is the required plane.

**Proof.** Const., $CD \perp AB$ in plane $MN$, and $DE \perp AB$ in plane $AF$; consequently, § 473, $\angle CDE$ is the plane $\angle$ of the dihedral $\angle N\text{-}AB\text{-}F$, and it is equal to the given $\angle$;

hence, § 479, dihedral $\angle N\text{-}AB\text{-}F =$ the given $\angle$.

**Ex. 143.** Through a given line parallel to a given plane pass a plane to make a given angle with the given plane.



**Solution.** Pass a plane $RS \perp AB$, the given line, at any point, as $C$, intersecting the given plane $MN$ as in $RE$; in the plane $RS$ draw $CD$ making $\angle CDE =$ the given $\angle$.

Through $AB$ and $CD$ pass a plane $PQ$ intersecting $MN$ as in $FP$.

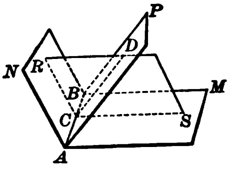
Then, $PQ$ is the plane required.

**Proof.** § 458, $FP \parallel AB$; but $AB \perp RS$;

$\therefore$ § 454, $FP \perp RS$; hence, § 443, $CD$ and $DE \perp FP$,

and, § 473, $\angle CDE$ is the plane $\angle$ of the dihedral $\angle$ between planes $PQ$ and $MN$; but $\angle CDE =$ the given $\angle$; hence, § 479, $PQ$ makes the given $\angle$ with $MN$.

**Ex. 144.** Through the edge of a given dihedral angle pass a plane to bisect that angle.



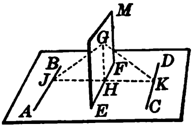
**Solution.** Pass a plane $RS \perp AB$, the edge of the given dihedral $\angle M\text{-}AB\text{-}N$, at any point $C$, intersecting the faces $BN$ and $AM$ in $CR$ and $CS$ respectively.

In $RS$ draw $CD$ to bisect $\angle RCS$ and pass a plane $AP$ through $CD$ and $AB$.

Then, $AP$ is the plane required.

**Proof.** Since, § 443, $CR$, $CD$, and $CS \perp AB$, § 473, $\angle$s $RCD$ and $DCS$ are the plane $\angle$s of the dihedral $\angle$s $N\text{-}AB\text{-}P$, and $P\text{-}AB\text{-}M$ respectively, and since, const., $\angle RCD = \angle DCS$, § 479, $\angle N\text{-}AB\text{-}P = \angle P\text{-}AB\text{-}M$; that is, plane $AP$ bisects dihedral $\angle M\text{-}AB\text{-}N$.

Ex. **145.** Find the locus of the points in space which are equidistant from two parallel lines.



**Solution.** Since, Ex. 371, the locus of the points equidistant from the given lines $AB$ and $CD$, in their plane, is the line $EF \parallel AB$ and $CD$ and midway between them, the locus of the points in space equidistant from $AB$ and $CD$ is the plane $EM \perp$ plane of $AB$ and $CD$ through $EF$.

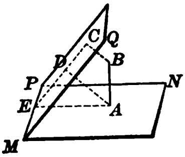
**Proof.** From any point $G$ in $EM$ draw $GH \perp EF$; draw $HJ \perp AB$, $HK \perp CD$, and draw $GJ$ and $GK$. Then, since, § 480, $GH \perp$ plane $AD$, § 451, $GJ \perp AB$, and $GK \perp CD$; and, since, Ex. 371, $HJ = HK$, § 447, $GJ = GK$; that is, § 62, $G$ is equidistant from $AB$ and $CD$.

Since, § 72, $HJ$ and $HK \perp EF$, § 58, $JHK$ is a straight line, and a plane may be passed through $JKG$.

Since, § 103, any point in the plane $JKG$ not in $GH$ is unequally distant from $J$ and $K$; that is, any point in plane $JKG$ not in plane $EM$ is unequally distant from $AB$ and $CD$, and since $G$ is any point in $EM$, the plane $JKG$ may have any position parallel to itself; consequently, any point, not in $EM$, is unequally distant from $AB$ and $CD$.

Hence, the locus of the points in space, which are equidistant from two parallel lines, is the plane perpendicular to their plane through the line midway between them.

Ex. **146.** If a straight line is perpendicular to a plane, its projection on any other plane is perpendicular to the intersection of the two planes.



**Data:** $AB \perp$ plane $MN$ and the projection $CD$ of $AB$ on plane $PQ$ which intersects $MN$ in $MP$.

**To prove** $CD \perp MP$.

**Proof.** Produce $CD$ to meet $MP$ as at $E$, and pass a plane through $CE$ and $AB$, intersecting $MN$ as in $AE$.

Then, § 482, plane $ABCE \perp MN$ and $PQ$;

hence, § 485, plane $ABCE \perp MP$, and, § 443, $MP \perp CE$;

that is, $CD \perp MP$.

Ex. **147.*** If two planes are perpendicular, a straight line drawn from any point of one plane perpendicular to the other will lie in the first plane.

**Data:** Planes $MN$ and $PQ$ perpendicular to each other and intersecting in $QD$; also the straight line $AB$ drawn $\perp MN$ from a point $A$ in $PQ$.

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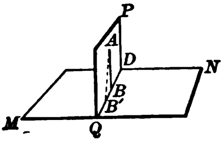
* See page 312.

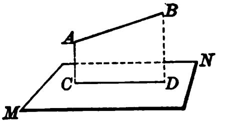


**To prove** $AB$ lies in $PQ$.

**Proof.** In $PQ$ draw $AB' \perp QD$.

Then, § 480, $AB' \perp MN$; but, data, $AB \perp MN$;
consequently, § 456, $AB$ and $AB'$ coincide;
but, const., $AB'$ lies in $PQ$;
hence, $AB$ lies in $PQ$.

Ex. **148.*** The projection of a straight line upon a plane is a straight line.



**Data:** The projection $CD$ of the straight line $AB$ upon the plane $MN$.

**To prove** $CD$ a straight line.

**Proof.** Through $AB$ pass a plane perpendicular to $MN$.

Then, Ex. 147, Rev., the perpendiculars drawn from the several points of $AB$ to $MN$ lie in this plane, and their feet lie in the intersection of this plane with $MN$; hence, §§ 437, 438, $CD$ must be the intersection of $MN$ and the plane passed through $AB$; therefore, § 441, $CD$ is a straight line.

Ex. **149**. Find the locus of the points in space which are equidistant from three given planes.

**Solution.** Since, Ex. 765, every point within a dihedral and equidistant from its faces lies in the plane which bisects that dihedral, the intersection of the planes bisecting any two of the dihedrals formed by the given planes is the locus of the points equidistant from them.

**Sch.** If the given planes intersect in a single point the locus passes through that point.

If two of the planes are parallel, the locus is parallel to the intersections of the given planes.

If all the given planes are parallel or intersect in a single line, there is no solution.

Ex. **150.** Find the locus of the points in space equidistant from two intersecting straight lines.

**Solution.** Since, Ex. 369, the lines $EF$ and $GH$, bisecting the vertical $\angle$s formed by the given lines $AB$ and $CD$, constitute the locus of the points equidistant from $AB$ and $CD$, in their plane, the planes through $EF$ and $GH$ perpendicular to the plane of $AB$ and $CD$ constitute the locus of the points in space equidistant from $AB$ and $CD$.

**Proof.** Proceed as in Ex. 145, Rev.

Ex. **151**. Two trihedrals are equal or symmetrical, if two face angles, and the dihedral between their faces, in one are equal, each to each, to the corresponding parts in the other.

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* See page 313.

**Proof.** If the corresponding parts are in the same order, apply one trihedral to the other so that their vertices and the given equal dihedrals shall coincide; then, the given equal face $\angle$s coincide each to each; consequently, the third face $\angle$ of one coincides with the third face $\angle$ of the other; hence, § 500, the trihedrals are equal.

If the corresponding parts are in the reverse order in the two trihedrals, then, one may be made to coincide with the symmetrical of the other.

Consequently, the trihedrals are equal or symmetrical.

**Ex. 152.** Find the locus of the points in space which are equidistant from three given straight lines in the same plane.

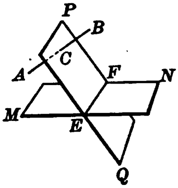
**Solution.** Since, if no two of the given lines are parallel, and they do not intersect each other, they may be produced to intersect and form a △, and since the centers of the inscribed and escribed Ⓢ of this △ are each equidistant from the given lines, the four lines perpendicular to the plane of the given lines at these centers constitute the required locus.

**Proof.** Show that points in these ⊥s are equidistant from the given lines, as in Ex. 145, Rev.

**Sch.** If two of the given lines are parallel, there are only two ⊥s constituting the required locus, and these are perpendicular to the plane of the given lines at the centers of the two Ⓢ, which are tangent to the given lines. If the given lines are all parallel, there is no solution.

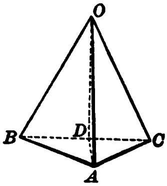
**Ex. 153.** Find the locus of the points in a given plane which are equidistant from two given points without the plane.

**Solution.** Since, Ex. 757, the plane $PQ \perp AB$, the line joining the given points, at its middle point $C$, is the locus of all points in space equidistant from $A$ and $B$, the intersection $EF$ of $PQ$ with the given plane $MN$ is the required locus.



**Ex. 154.** The angles $AOB$ and $AOC$ in different planes are equal. Prove that the plane bisecting the dihedral angle between their planes is perpendicular to the plane $BOC$.

**Proof.** Let $AOD$ be the plane bisecting dihedral $AO$. In trihedrals $O\text{-}ADB$ and $O\text{-}ADC$, face $\angle AOD$ is common, data, face $\angle AOB =$ face $\angle AOC$, dihedral $\angle B\text{-}AO\text{-}D =$ dihedral $\angle C\text{-}AO\text{-}D$, and since the corresponding parts are in reverse order, Ex. 151, Rev., trihedrals $O\text{-}ADB$ and $O\text{-}ADC$ are symmetrical; hence, dihedral $\angle A\text{-}OD\text{-}B =$ dihedral $\angle A\text{-}OD\text{-}C$; consequently, § 475, each of these dihedrals is a right dihedral, and plane $AOD \perp$ plane $BOC$.



**Ex. 155.** The planes through any two pairs of lines that pass through a point intersect in a line which passes through the same point.

**Proof.** Since the given point is common to the two planes, § 436, it must be in their intersection; that is, the intersection passes through the point common to the given lines.

**Ex. 156.** In a given plane, find a point equidistant from three given points without the plane.

**Solution.** Through the given points, pass the circumference of a ⊙, and through the center of the ⊙, draw a line perpendicular to the plane of the ⊙, producing it to intersect the given plane.

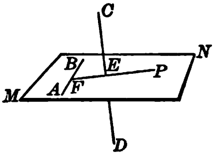
Then, § 450, the point of intersection is the point required.

**Ex. 157.** Through a given point in space, draw a straight line which shall cut two given straight lines not in the same plane.

**Solution.** Through the given point $P$ and one of the given lines $AB$, pass a plane $MN$ intersecting the other given line $CD$ in a point, as $E$.

Draw $PE$ and produce it (if necessary) to intersect $AB$ as in $F$.

Then, $PF$ is the required line.

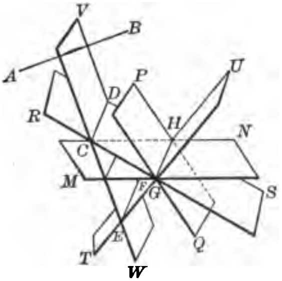


**Proof.** Since $AB$ and $CD$ are not in the same plane, they are not parallel, and the plane of $P$ and $AB$ must cut $CD$, or $CD$ produced in some point $E$, and, since $E$ and $P$ are in $MN$, $PEF$ is in $MN$, and must intersect $AB$.

**Sch.** $E$ may be so situated that $PE \parallel AB$, in which case there is no solution.

**Ex. 158.** Find the locus of the points in space which are equidistant from two given planes and also equidistant from two given points.

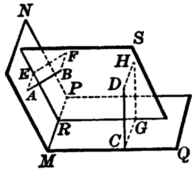
**Solution.** Since, Ex. 765, every point equidistant from the given planes $MN$ and $PQ$ is in the planes $RS$ and $TU$ bisecting the vertical dihedrals contained by them, and since, Ex. 757, the locus of all points in space equidistant from the given points $A$ and $B$ is the plane $VW$ perpendicular to the line $AB$ at its middle point, the intersections $CD$ and $EF$ of plane $VW$ with planes $RS$ and $TU$ constitute the required locus.



**Sch.** If $AB \perp RS$ or $TU$, there is only one line in the locus.

If $MN \parallel PQ$, the required locus consists of one line, which is the intersection of plane $VW$ with, Ex. 753, the plane parallel to $MN$ and $PQ$ and midway between them; but in this case there is no solution if $AB \perp MN$ and $PQ$.

**Ex. 159.** Two planes are perpendicular respectively to two non-parallel lines which are not in the same plane. Prove that their intersection is perpendicular to any plane that is parallel to both lines.



**Data:** Planes $MN$ and $PQ$ perpendicular respectively to lines $AB$ and $CD$ and intersecting in $MP$; also any plane, as $RS \parallel AB$ and $CD$.

**To prove** $MP \perp RS$.

**Proof.** Through $AB$ pass a plane $AF$ intersecting $RS$ as in $EF$; also through $CD$ pass a plane $CH$ intersecting $RS$ as in $GH$.

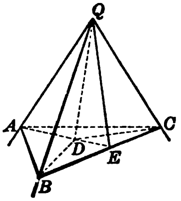
Then, § 458, $EF \parallel AB$, and $GH \parallel CD$; $\therefore$ $EF \perp MN$, and $GH \perp PQ$;

hence, § 482, $RS \perp MN$ and $PQ$;

consequently, § 485, $RS \perp MP$; that is, $MP \perp RS$.

**Ex. 160.** From the vertex of a trihedral angle a line is drawn within the angle. Prove that the sum of the angles which this line makes with the edges is less than the sum, but greater than half the sum, of the face angles.

**Data:** Trihedral $Q\text{-}ABC$, and $QD$ a line within drawn from the vertex $Q$.



**To prove** 1. $\angle AQD + \angle BQD + \angle CQD < \angle AQB + \angle BQC + \angle AQC.$

2. $\angle AQD + \angle BQD + \angle CQD > \frac{1}{2}(\angle AQB + \angle BQC + \angle AQC).$

**Proof.** 1. Produce the plane of $AQD$ to intersect $BQC$ as in $QE$.

Then, § 498, $\angle AQC + \angle CQE > \angle AQD + \angle DQE,$

and $\angle DQE + \angle EQB > \angle BQD.$

Adding these inequalities and taking $\angle DQE$ from each member,

$$\angle AQC + \angle CQE + \angle EQB > \angle AQD + \angle BQD,$$

or $\angle AQC + \angle BQC > \angle AQD + \angle BQD.$

Similarly, $\angle AQB + \angle BQC > \angle AQD + \angle CQD,$

and $\angle AQB + \angle AQC > \angle BQD + \angle CQD;$

$\therefore$ adding,

$$2\angle AQB + 2\angle BQC + 2\angle AQC > 2\angle AQD + 2\angle BQD + 2\angle CQD,$$

or $\angle AQD + \angle BQD + \angle CQD < \angle AQB + \angle BQC + \angle AQC.$

2. § 498, $\angle AQD + \angle BQD > \angle AQB,$

$\angle BQD + \angle CQD > \angle BQC,$

and $\angle AQD + \angle CQD > \angle AQC;$

$\therefore$ adding, $2\angle AQD + 2\angle BQD + 2\angle CQD > \angle AQB + \angle BQC + \angle AQC,$

or $\angle AQD + \angle BQD + \angle CQD > \frac{1}{2}(\angle AQB + \angle BQC + \angle AQC).$

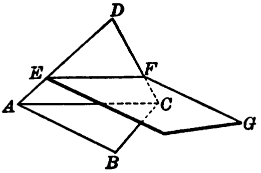
**Ex. 161.** Two trihedrals are equal or symmetrical, if two dihedrals and the included face angles of one are equal, each to each, to the corresponding parts of the other.

**Proof.** If the corresponding parts are in the same order in the two trihedrals, apply one to the other so that the equal face ∠s shall coincide; then, since the including dihedrals are respectively equal, the planes of the other two face ∠s of one trihedral take the same directions respectively as the planes of the corresponding face ∠s of the other trihedral, and, consequently, the intersections of these planes coincide; that is, the third edge of one trihedral coincides with the third edge of the other; hence, the trihedrals coincide throughout and are equal.

If the corresponding parts are in reverse order in the two trihedrals, then, one may be made to coincide with the symmetrical of the other.

Consequently, the trihedrals are equal or symmetrical.

**Ex. 162.** A plane parallel to two sides of a quadrilateral in space (that is, a quadrilateral whose sides do not all lie in the same plane) divides the other two sides proportionally.



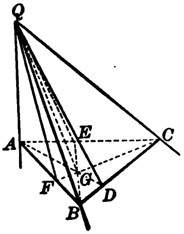
**Data:** Quad. in space, as $ABCD$, whose sides $AB$, $BC$ and $AD$, $DC$ are in different planes; also plane $EG$ parallel to $AB$, $BC$, and intersecting plane $ACD$, as in $EF$.

**To prove** $AE:ED = CF:FD$.

**Proof.** § 463, $EF \parallel AC$;

∴ § 289, $AE:ED = CF:FD$.

**Ex. 163.** In any trihedral, the three planes passed through the edges perpendicular to the opposite faces, respectively, intersect in the same straight line.



**Data:** Trihedral $Q$-$ABC$ and the planes through the edges $QA$, $QB$, and $QC$ perpendicular to faces $BCQ$, $ACQ$, and $ABQ$ respectively.

**To prove** these planes intersect in the same straight line.

**Proof.** At any point in the edge $QA$, as $A$, pass a plane $\perp QA$ forming by its intersections with the faces of the trihedral the $\triangle ABC$ and intersecting the planes through $QA$, $QB$, and $QC$ in $AD$, $BE$, and $CF$ respectively.

Then, § 482, plane $ADQ \perp$ plane $ABC$, and, data, plane $ADQ \perp$ plane $BCQ$;

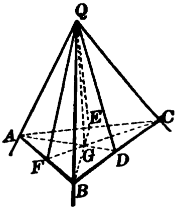
∴ § 485, plane $ADQ \perp BC$, and, § 443, $AD \perp BC$;

§ 482, $ACQ \perp ABC$; that is, $ABC \perp ACQ$, data, $BEQ \perp ACQ$;

∴ § 484, $BE \perp ACQ$, and, § 443, $BE \perp AC$;

similarly, $CF \perp AB$;
hence, § 171, $AD$, $BE$, and $CF$ intersect at $G$; $\therefore$ $Q$ and $G$ are common to the three planes; consequently, the planes intersect in the line $QG$.

**Ex. 164.** In any trihedral, the three planes, passed through the edges and the bisectors of the opposite face angles, respectively, intersect in the same straight line.



**Data:** Trihedral $Q$-$ABC$ and the planes through its edges $QA$, $QB$, $QC$, and the bisectors $QD$, $QE$, $QF$ of the opposite face $\angle$s respectively.

**To prove** these planes intersect in the same straight line.

**Proof.** Take $QA = QB = QC$ and through $A$, $B$, and $C$ pass a plane forming by its intersection with the faces $\triangle ABC$ and cutting $QD$, $QE$, $QF$ at $D$, $E$, $F$ respectively.

Then, $\triangle$s $QAB$, $QBC$, and $QAC$ are isosceles; and, § 120, $F$, $D$, $E$ are the middle points of $AB$, $BC$, $AC$, respectively, and the planes passed through the edges and the bisectors of the opposite face angles intersect $\triangle ABC$ in the medians $AD$, $BE$, and $CF$; but, § 168, $AD$, $BE$, and $CF$ intersect in a point $G$; hence, $Q$ and $G$ are common to the three planes; consequently, the planes intersect in the line $QG$.

**Ex. 165.** What is the edge of a cubical vessel that holds one half ton of water?

**Solution.** Since a cubic foot of water weighs $62\frac{1}{2}$ lb., the vol. of $\frac{1}{2}$ ton of water $= 1000 \div 62\frac{1}{2} = 16$ cu. ft.; hence, the edge of the vessel $= \sqrt[3]{16} = 2.5193+$ ft.

**Ex. 166.** Represent the base edge of a regular four-sided pyramid by $e$, its altitude by $h$, and its total surface by $T$. Compute the base edge in terms of $h$ and $T$.

**Solution.** Slant height $= \sqrt{(\frac{1}{2}e)^2 + h^2} = \frac{1}{2}\sqrt{e^2 + 4h^2}$;

hence, § 551, lateral surface $= \frac{1}{2} \times 4e \times \frac{1}{2}\sqrt{e^2 + 4h^2} = e\sqrt{e^2 + 4h^2}$;

but lateral surface $= T - e^2$;

hence, $e\sqrt{e^2 + 4h^2} = T - e^2$;

squaring, $e^2(e^2 + 4h^2) = T^2 - 2Te^2 + e^4$;

whence, $e^2 = \dfrac{T^2}{2T + 4h^2}$, and $e = \dfrac{T}{\sqrt{2T + 4h^2}}$.

**Ex. 167.** What is the difference in volume between the frustum of a pyramid and a prism each $12^{dm}$ high, if the bases of the frustum are squares whose sides are $10^{dm}$ and $8^{dm}$ respectively, and the base of the prism is a section of the frustum parallel to its bases and midway between them?

**Solution.** § 564, vol. of frustum $= \frac{1}{3} \times 12(10^2 + 8^2 + \sqrt{10^2 \times 8^2}) = 976^{\text{cu dm}}$.

§ 160, side of base of prism $= \frac{1}{2}(10^{\text{dm}} + 8^{\text{dm}}) = 9^{\text{dm}}$;

hence, § 543, vol. of prism $= 9^2 \times 12 = 972^{\text{cu dm}}$.

Hence, the difference in vol. of prism and frustum is

$$976^{\text{cu dm}} - 972^{\text{cu dm}} = 4^{\text{cu dm}}.$$

**Ex. 168.** What is the volume of a regular tetrahedron whose edge is $10^{\text{dm}}$?

**Solution.** Ex. 826, vol. $= 10^3 \times \frac{1}{12}\sqrt{2} = 117.85+^{\text{ cu dm}}$.

**Ex. 169.** The altitude of a regular hexagonal pyramid is 13 in., and its slant height is 16 in. What is its lateral edge?

**Solution.** Apothem of base $= \sqrt{16^2 - 13^2} = \sqrt{87}$;

but, Ex. 648, apothem $= \frac{1}{2} r\sqrt{3}$; hence, $\frac{1}{2} r\sqrt{3} = \sqrt{87}$, and $r = \frac{2\sqrt{87}}{\sqrt{3}} = 2\sqrt{29}$.

Then, lateral edge $= \sqrt{13^2 + (2\sqrt{29})^2} = \sqrt{169 + 116} = \sqrt{285} = 16.88+$ in.

**Ex. 170.** The lateral faces of a regular quadrangular pyramid are equilateral triangles, and its altitude is $9^{\text{m}}$. What is the area of the base?

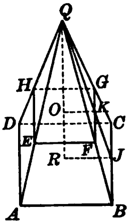
**Solution.** Denote the side of the base by $x$.

Then, § 351, diagonal of base $= x\sqrt{2}$, and $\frac{1}{2}$ diag. $= \frac{x}{\sqrt{2}}$.

Since lateral edge $= x$, $x^2 = \left(\frac{x}{\sqrt{2}}\right)^2 + 9^2$; that is, $x^2 = \frac{x^2}{2} + 81$;

whence, $x^2 = 162$; that is, area of base $= 162^{\text{sq m}}$.

**Ex. 171.** The altitude of a frustum of a regular quadrangular pyramid is $10^{\text{cm}}$, and the sides of its bases are respectively $16^{\text{cm}}$ and $6^{\text{cm}}$. What is the lateral area of the frustum?



**Solution.** Let $EFGH - ABCD$ be the frustum of a regular quadrangular pyramid $Q\text{-}ABCD$, the altitude $OR$ of the frustum being $10^{\text{cm}}$ and the sides of its bases $16^{\text{cm}}$ and $6^{\text{cm}}$ respectively. Then, § 299,

$QO : QO + OR = OK : RJ$; that is, $QO : QO + 10 = 3 : 8$;

whence, $QO = 6^{\text{cm}}$.

§ 349, $QK = \sqrt{\overline{QO}^2 + \overline{OK}^2} = \sqrt{36 + 9} = \sqrt{45} = 3\sqrt{5}$;

then, § 289, $QO : OR = QK : KJ$; that is, $6 : 10 = 3\sqrt{5} : KJ$;

whence, $KJ = \frac{10 \times 3\sqrt{5}}{6} = 5\sqrt{5} = 11.18+^{\text{ cm}}$.

§§ 583, 552, $A = \frac{1}{2} L(P + P') = \frac{1}{2} \times 11.18(64 + 24) = 491.92+^{\text{ sq cm}}$.

**Ex. 172.** If the altitude of a pyramid is $h$, at what distance from the vertex will it be cut by a plane parallel to the base and dividing the pyramid into two parts which are in the ratio of 3 : 4?

**Solution.** Ex. 817, the pyramid cut off is similar to the original pyramid, and these pyramids are in the ratio of 3 : 7.

Then, denoting the altitude of the small pyramid by $h'$,

§§ 576, 574, $$V' : V = h'^3 : h^3;$$

but $$V' : V = 3 : 7; \text{ hence, } h'^3 : h^3 = 3 : 7;$$

whence, $$h'^3 = \frac{3h^3}{7}, \text{ and } h' = h\sqrt[3]{\tfrac{3}{7}};$$

that is, the plane cuts the altitude at the distance $h\sqrt[3]{\tfrac{3}{7}}$ from the vertex.

**Ex. 173.** At what distances from the vertex will a lateral edge of a pyramid be cut by two planes parallel to the base, if they divide the pyramid into three equivalent parts, the length of the edge being $m$?

**Solution.** Denote the distance on an edge from the vertex to the intersection of the plane nearest to it by $x$.

Then, Ex. 817, § 576, $$V : V' = m^3 : x^3;$$

but $$V : V' = 3 : 1; \text{ hence, } m^3 : x^3 = 3 : 1;$$

whence, $$x^3 = \frac{m^3}{3}, \text{ and } x = m\sqrt[3]{\tfrac{1}{3}}.$$

Similarly, denoting the distance to the intersection nearest the base by $y$,

$$m^3 : y^3 = 3 : 2; \text{ whence, } y^3 = \frac{2m^3}{3}, \text{ and } y = m\sqrt[3]{\tfrac{2}{3}}.$$

**Ex. 174.** If the base edge of a regular square pyramid is $m$, and its total surface is $T$, what is its volume?

**Solution.** §§ 551, 583, $A = \frac{1}{2} \times 4m \times L = 2mL$;

but $$A = T - m^2; \text{ hence, } 2mL = T - m^2;$$

whence, $$L = \frac{T - m^2}{2m};$$

and since apothem of base $= \dfrac{m}{2}$,

$$H = \sqrt{\left(\frac{T - m^2}{2m}\right)^2 - \left(\frac{m}{2}\right)^2} = \sqrt{\frac{T^2 - 2Tm^2 + m^4}{4m^2} - \frac{m^2}{4}} = \frac{\sqrt{T^2 - 2Tm^2}}{2m},$$

and since area of base $= m^2$,

§ 560, $$V = \frac{1}{3} \times m^2 \times \frac{\sqrt{T^2 - 2Tm^2}}{2m} = \frac{m}{6}\sqrt{T^2 - 2Tm^2}.$$

**Ex. 175.** The perimeter of the base of a regular quadrangular pyramid is $p$, and the area of a section through two diagonally opposite edges is $A$. What is the lateral area of the pyramid?

**Solution.** § 351, the base of the section or the diagonal of the base of the pyramid $= \frac{p}{4}\sqrt{2}$;

hence, $A = \frac{1}{2} \times H \times \frac{p}{4}\sqrt{2} = \frac{Hp\sqrt{2}}{8}$; whence, $H = \frac{8A}{p\sqrt{2}}$,

and since apothem of base $= \frac{1}{2}$ of side $= \frac{p}{8}$,

$$L = \sqrt{\left(\frac{8A}{p\sqrt{2}}\right)^2 + \left(\frac{p}{8}\right)^2} = \sqrt{\frac{64A^2}{2p^2} + \frac{p^2}{64}} = \frac{1}{8p}\sqrt{2048A^2 + p^4}.$$

Hence, $\text{lat. area} = \frac{1}{2} \times p \times \frac{1}{8p}\sqrt{2048A^2 + p^4} = \frac{\sqrt{2048A^2 + p^4}}{16}.$

Ex. **176**. If two tetrahedrons have the faces including a trihedral of one similar to the faces including a trihedral of the other, each to each, and similarly placed, the tetrahedrons are similar.

**Data**: (See Fig. for § 575.) Tetrahedrons *Q-ABC* and *T-DEF*, having faces *ABQ*, *BCQ*, and *ACQ*, including trihedral $\angle Q$, similar respectively to faces *DET*, *EFT*, and *DFT*, including trihedral $\angle T$.

**To prove** *Q-ABC* and *T-DEF* similar.

**Proof.** From given similar faces,

§ 299, $AB : DE = (BQ : ET) = BC : EF = (CQ : FT) = AC : DF$;

hence, § 303, faces *ABC* and *DEF* are similar, and, § 299, the face $\triangle$s including trihedral $\angle A$ are equal respectively to the face $\triangle$s including trihedral $\angle D$; hence, § 500, trihedral $\angle A$ = trihedral $\angle D$.

Similarly, any two homologous trihedrals are equal.

Hence, § 569, *Q-ABC* and *T-DEF* are similar.

Ex. **177**. If two tetrahedrons have a dihedral angle of one equal to a dihedral angle of the other, and the faces including these dihedrals similar, each to each, and similarly placed, the tetrahedrons are similar.

**Data**: (See Fig. for § 575.) Tetrahedrons *Q-ABC* and *T-DEF* having dihedral $\angle AQ$ = dihedral $\angle DT$, and faces *ABQ* and *ACQ* similar to *DET* and *DFT*, respectively.

**To prove** *Q-ABC* and *T-DEF* similar.

**Proof.** Apply *T-DEF* to *Q-ABC* so that dihedral *TD* shall coincide with its equal *QA*, the point *T* falling at *Q*.

Since, § 299, $\angle DTE = \angle AQB$, and $\angle DTF = \angle AQC$, *TE* will take the direction of *QB* and *TF* will take the direction of *QC*;

hence, $\angle BQC = \angle ETF$,

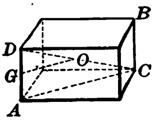
and from the given similar faces,

§ 299, $QB : TE = (QA : TD) = QC : TF$;

hence, § 306, faces *BCQ* and *EFT* are similar, and the faces including trihedral $\angle Q$ are similar to the faces including trihedral $\angle T$.

Hence, Ex. 176, Rev., *Q-ABC* and *T-DEF* are similar.

**Ex. 178.** The perpendicular from the middle point of the diagonal of a rectangular parallelopiped upon a lateral edge bisects the edge and is equal to one half the projection of the diagonal upon the base.



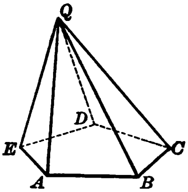
**Data**: Rectangular parallelopiped $AB$; diagonal $CD$; its projection $CA$ on the base; and $OG$, from $O$ the middle point of $CD$, perpendicular to $AD$, a lateral edge.

**To prove** $AG = DG$ and $OG = \frac{1}{2} CA$.

**Proof.** § 443, $CA \perp AD$ in plane $ACD$, and, data, $OG \perp AD$ in plane $ACD$; $\therefore$ § 71, $OG \parallel CA$; and, data, $CO = DO$; hence, § 158, $AG = DG$, and $OG = \frac{1}{2} CA$.

The same is true for the $\perp$ from the middle point of diagonal $AB$ to edge $BC$, and since the diagonals bisect each other at $O$ the same is true for the $\perp$ from $O$ to *any* lateral edge.

**Ex. 179.** In any polyhedron the number of edges increased by two is equal to the number of vertices increased by the number of faces.



**Data**: Polyhedron $Q$-$ABCDE$ having $E$ edges, $V$ vertices, and $F$ faces.

**To prove** $E + 2 = V + F$.

**Proof.** Any face, as $ABCDE$, has as many edges as vertices; that is, for one face $E = V$.

If a second face $QAB$ is annexed by applying one of its edges $AB$ to the corresponding edge of the first face, there is formed a surface having one edge $AB$ and two vertices $A$ and $B$ common to the two faces; consequently, the whole number of edges is now one more than the whole number of vertices;

that is, for two faces, $E = V + 1$.

If a third face $QBC$ is annexed, adjacent to each of the first two faces, it will have two edges, $QB$, $BC$, and three vertices $Q$, $B$, $C$ in common with the surface already formed;

that is, for three faces, $E = V + 2$.

Similarly, for four faces, $E = V + 3$,

and for $(F - 1)$ faces, $E = V + (F - 2)$.

Now, $F - 1$ is the number of faces of the polyhedron when one face is lacking, and by annexing this face the number of edges or vertices will not be increased.

Hence, for $F$ faces, $E = V + F - 2$;

that is $E + 2 = V + F$.

**Ex. 180.** The sum of the face angles of any polyhedron is equal to four right angles taken as many times, less two, as the polyhedron has vertices.

**Data**: (See Fig. for Ex. 179, Rev.) Any polyhedron as $Q$-$ABCDE$ having $E$ edges, $F$ faces, and $V$ vertices.

Denote the sum of its face $\angle$s by $S$.

**To prove** $S=(V-2)\,4$ rt. $\angle$s.

**Proof.** Since each edge is common to two faces, the whole number of sides of the faces considered as separate polygons is $2E$.

Form an exterior $\angle$ at each vertex of each polygon; then, the sum of the interior and exterior $\angle$s at each vertex is 2 rt. $\angle$s, and since there are $2E$ vertices the sum of all the interior and exterior $\angle$s of the faces is $2E \times 2$ rt. $\angle$s, or $E \times 4$ rt. $\angle$s; but, § 167, the sum of the exterior $\angle$s of each face is 4 rt. $\angle$s; $\therefore$ the sum of the exterior $\angle$s of the $F$ faces is $F \times 4$ rt. $\angle$s;

hence, $S = E \times 4$ rt. $\angle$s $- F \times 4$ rt. $\angle$s $=(E-F)\,4$ rt. $\angle$s.

But, Ex. 179, Rev., $E+2=V+F$; whence, $E-F=V-2$;

hence, $S=(V-2)\,4$ rt. $\angle$s.

**Ex. 181.** If a plane is tangent to a circular cone, its intersection with the plane of the base is tangent to the base.

**Proof.** Since, § 615, the tangent plane does not cut the surface of the cone, it does not cut the circumference of the base of the cone, and, § 441, its intersection with the plane of the base cannot cut the circumference of the base.

Since the tangent plane contains an element of the cone, one extremity of the element is common to the circumference of the base, the tangent plane, and hence the intersection of the tangent plane with the plane of the base; consequently, § 184, this intersection is tangent to the base.

**Ex. 182.*** If a plane is tangent to a circular cylinder, its intersection with the plane of the base is tangent to the base.

**Proof.** § 441, the intersection of the tangent plane with the plane of the base is a straight line, and since, § 591, the tangent plane does not cut the surface of the cylinder, it does not cut the circumference of the base of the cylinder; consequently, the line in which it intersects the plane of the base does not cut the circumference of the base; but it touches the circumference at the extremity of the element of contact; hence, § 184, the intersection of the tangent plane with the plane of the base is tangent to the base.

**Ex. 183.** What are the dimensions of a cylindrical measure whose altitude is half its diameter, if it holds a half bushel?

**Solution.** Data,

$H=\frac{1}{2}D=R$, and vol. $=\frac{1}{2}$ of 2150.42 cu. in., or 1075.21 cu. in.;

but vol. $=\pi R^2H=\pi R^3$;

hence, $\pi R^3=1075.21$; whence, $R^3=\dfrac{1075.21}{3.1416}=342.249172$,

and $R=\sqrt[3]{342.249172}=7$ in. nearly, and diameter $=14$ in. nearly;

hence, altitude $=7$ in. nearly.

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* See page 313.

**Ex. 184.** Find the weight of the water that will be contained in a vertical pipe 40 ft. high and 1 ft. in diameter. Also find the pressure per square inch on the base of the pipe.

**Solution.** § 606, $V = \pi R^2 H = \pi\left(\frac{1}{2}\right)^2 40 = 31.416$ cu. ft., and since a cu. ft. of water weighs $62\frac{1}{2}$ lb., the pipe will contain $62\frac{1}{2}$ lb. $\times\ 31.416 = 1963.5$ lb.

Since area of base $= \pi R^2 = 3.1416 \times 6^2 = 113.0976$ sq. in., the pressure on each sq. in. is $1963.5 \div 113.0976 = 17.36+$ lb.

**Ex. 185.** A rectangle revolves successively about two adjacent sides whose lengths are $m$ and $n$ respectively. Compare the volumes of the cylinders thus generated.

**Solution.** § 606, vol. of first $= \pi R^2 H = \pi n^2 m$, and vol. of second $= \pi m^2 n$; hence, vol. of first : vol. of second $= \pi n^2 m : \pi m^2 n = n : m$.

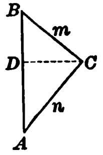
**Ex. 186.** A right triangle revolves successively about the perpendicular sides whose lengths are $m$ and $n$ respectively. Compare the volumes of the cones thus generated.

**Solution.** § 630, vol. of first $= \frac{1}{3}\pi R^2 H = \frac{1}{3}\pi n^2 m$,

and vol. of second $= \frac{1}{3}\pi m^2 n$;

hence, vol. of first : vol. of second $= \frac{1}{3}\pi n^2 m : \frac{1}{3}\pi m^2 n = n : m$.

**Ex. 187.** If the sides including the right angle of a right triangle are $m$ and $n$, what is the area of the surface generated by revolving the triangle about its hypotenuse as an axis.



**Solution.** § 313, $AB : m = m : BD$; and $AB : n = n : AD$;

whence, $$BD = \frac{m^2}{AB} = \frac{m^2}{\sqrt{m^2+n^2}};$$

and $$AD = \frac{n^2}{AB} = \frac{n^2}{\sqrt{m^2+n^2}}.$$

§ 312, $$BD : CD = CD : AD;$$

that is, $$\overline{CD}^2 = BD \times AD = \frac{m^2}{\sqrt{m^2+n^2}} \times \frac{n^2}{\sqrt{m^2+n^2}} = \frac{m^2n^2}{m^2+n^2},$$

and $$CD = \frac{mn}{\sqrt{m^2+n^2}}.$$

Then, § 624, surf. generated by $BC = \pi RL = \pi CD \times L$

$$= \pi\frac{mn}{\sqrt{m^2+n^2}} \times m = \frac{\pi m^2 n}{\sqrt{m^2+n^2}};$$

similarly, surf. generated by $AC = \pi RL = \pi CD \times L$

$$= \pi\frac{mn}{\sqrt{m^2+n^2}} \times n = \frac{\pi m n^2}{\sqrt{m^2+n^2}};$$

consequently, whole surf. generated $= \dfrac{\pi m^2 n}{\sqrt{m^2+n^2}} + \dfrac{\pi m n^2}{\sqrt{m^2+n^2}} = \dfrac{\pi mn(m+n)}{\sqrt{m^2+n^2}}$.

**Ex. 188.** Find the altitude of a cylinder of revolution of radius $r$, if the cylinder is equivalent to a rectangular parallelopiped whose dimensions are $l$, $m$, and $n$.

**Solution.** § 536, vol. parallelopiped $= lmn$,

§ 606, vol. cylinder $= \pi r^2 H$; but, data, $\pi r^2 H = lmn$;

whence, $$H = \frac{lmn}{\pi r^2}.$$

**Ex. 189.** Find the altitude of a cone of revolution of radius $r$, equivalent to a rectangular parallelopiped whose dimensions are $l$, $m$, and $n$.

**Solution.** § 536, vol. parallelopiped $= lmn$;

§ 630, vol. cone $= \frac{1}{3}\pi r^2 H$; but, data, $\frac{1}{3}\pi r^2 H = lmn$;

whence, $$H = \frac{3\,lmn}{\pi r^2}.$$

**Ex. 190.** The altitudes of two equivalent cylinders of revolution are in the ratio $a : b$. If the radius of one is $r$, what is the radius of the other?

**Solution.** Denote the altitudes of the cylinders by $H$ and $H'$ respectively and the radius of the second by $r'$.

Then, $V = \pi r^2 H$, and $V' = \pi r'^2 H'$; but the cylinders are equivalent; $\therefore\ \pi r'^2 H' = \pi r^2 H$; whence, $r'^2 = r^2 \frac{H}{H'}$; but, data, $\frac{H}{H'} = \frac{a}{b}$; hence, $r'^2 = r^2 \frac{a}{b}$;

whence, $$r' = \frac{r}{b}\sqrt{ab}.$$

**Ex. 191.** Find the altitude of a regular quadrangular prism whose base edge is $m$, the prism being equivalent to a cylinder of revolution whose altitude is $h$ and radius $r$.

**Solution.** Denote the altitude of the prism by $H$.

§ 606, vol. cylinder $= \pi r^2 h =$ vol. of prism;

but base of prism $= m^2$, and, § 543, vol. of prism $= m^2 H$;

hence, $m^2 H = \pi r^2 h$; whence, $H = \frac{\pi r^2 h}{m^2}$.

**Ex. 192.** How must the dimensions of a cylinder of revolution be increased in order to form a similar cylinder whose total surface shall be $n$ times that of the original cylinder?

**Solution.** Denote the radius and altitude of the required cylinder by $R'$ and $H'$ respectively.

Then, § 603, $T : T' = R^2 : R'^2$; but $T' = nT$; $\therefore\ T : nT = R^2 : R'^2$; hence, $R'^2 = R^2 n$, and $R' = R\sqrt{n}$. Similarly, $H' = H\sqrt{n}$.

Consequently, the dimensions of the given cylinder must be multiplied by $\sqrt{n}$, in order to form the required cylinder.

**Ex. 193.** How must the dimensions of a cylinder of revolution be increased in order to form a similar cylinder whose volume shall be $n$ times that of the original cylinder?

**Solution.** Denote the radius and altitude of the required cylinder by $R'$ and $H'$ respectively.

Then, § 607, $V : V' = R^3 : R'^3$; but, $V' = nV$;

∴ $V : nV = R^3 : R'^3$; hence, $R'^3 = R^3 n$, and $R' = R\sqrt[3]{n}$.

Similarly, $H' = H\sqrt[3]{n}$. Consequently, the dimensions of the given cylinder must be multiplied by $\sqrt[3]{n}$ in order to form the required cylinder.

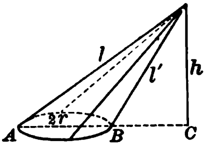
**Ex. 194.** What is the radius of the base of a circular cone whose altitude is $h$, the longest and the shortest elements being $l$ and $l'$ respectively?

**Solution.** § 350, $AC = \sqrt{l^2 - h^2}$,

and $BC = \sqrt{l'^2 - h^2}$;

then, $AB = 2r = \sqrt{l^2 - h^2} - \sqrt{l'^2 - h^2}$,

and $r = \dfrac{\sqrt{l^2 - h^2} - \sqrt{l'^2 - h^2}}{2}$.



**Ex. 195.** What is the slant height of a frustum of a cone of revolution whose lateral surface is $S$ and whose lower and upper bases are $B$ and $b$ respectively.

**Solution.** § 398, $B = \pi R^2$; whence, $R = \sqrt{\dfrac{B}{\pi}}$, and $C = 2\pi\sqrt{\dfrac{B}{\pi}} = 2\sqrt{\pi B}$, similarly, $C' = 2\sqrt{\pi b}$;

then, §§ 627, 635, $S = \frac{1}{2} L(2\sqrt{\pi B} + 2\sqrt{\pi b}) = L(\sqrt{\pi B} + \sqrt{\pi b})$;

whence, $L = \dfrac{S}{\sqrt{\pi B} + \sqrt{\pi b}}$.

**Ex. 196.** A cone of revolution whose radius is $r$ and altitude $h$ is divided into two equivalent parts by a plane parallel to the base. What is the total area of the frustum thus formed?

**Solution.** As in Ex. 844, the cone cut off is similar to the original cone.

Then, § 632, $V : V' = L^3 : L'^3$; but, $V : V' = 2 : 1$;

∴ $L^3 : L'^3 = 2 : 1$, or, § 288, $L : L' = \sqrt[3]{2} : 1$;

whence, $L' = \dfrac{L}{\sqrt[3]{2}} = \dfrac{\sqrt{r^2 + h^2}}{\sqrt[3]{2}}$;

consequently, the slant height of frustum $= L - L' = \sqrt{r^2 + h^2} - \dfrac{\sqrt{r^2 + h^2}}{\sqrt[3]{2}}$

$= \dfrac{\sqrt{r^2 + h^2}(\sqrt[3]{2} - 1)}{\sqrt[3]{2}}$.

Similarly, $r : r' = \sqrt[3]{2} : 1$; whence, $r' = \dfrac{r}{\sqrt[3]{2}}$.

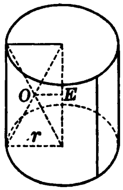
Then, §§ 627, 635, $A = \dfrac{1}{2}\,\dfrac{\sqrt{r^2 + h^2(\sqrt[3]{2} - 1)}}{\sqrt[3]{2}}\left(2\pi r + 2\pi\dfrac{r}{\sqrt[3]{2}}\right)$

$$= \frac{1}{2}\,\frac{\sqrt{r^2 + h^2(\sqrt[3]{2} - 1)}}{\sqrt[3]{2}}\left(\frac{2\pi r\sqrt[3]{2} + 2\pi r}{\sqrt[3]{2}}\right) = \frac{\pi r\sqrt{r^2 + h^2(\sqrt[3]{4} - 1)}}{\sqrt[3]{4}}.$$

Area of lower base $= \pi r^2$, and area of upper base $= \pi r'^2 = \pi\left(\dfrac{r}{\sqrt[3]{2}}\right)^2 = \dfrac{\pi r^2}{\sqrt[3]{4}}$.

Then, total area $= \dfrac{\pi r\sqrt{r^2 + h^2(\sqrt[3]{4} - 1)}}{\sqrt[3]{4}} + \pi r^2 + \dfrac{\pi r^2}{\sqrt[3]{4}}$.

**Ex. 197.** The volume of a cylinder of revolution is equal to the area of its generating rectangle multiplied by the circumference generated by the point of intersection of the diagonals of the rectangle.



**Proof.** Let $r$ denote the radius and $h$ the altitude of the cylinder, or the sides respectively of the generating rectangle, whose diagonals intersect at $O$.

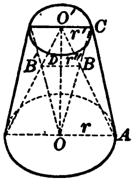
Then, area of rectangle $= rh$, and the circumference generated by the point $O = 2\pi OE = 2\pi\dfrac{r}{2} = \pi r$; hence, the product of the area of the rectangle and this circumference $= \pi r^2 h$; but, § 606, volume of cylinder $= \pi r^2 h$; hence, the volume of the cylinder is equal to the area of the generating rectangle multiplied by the circumference generated by $O$.

**Ex. 198.** On each base of the frustum of a cone of revolution there is a cone whose vertex is in the center of the other base. If the radii of the lower and upper bases are $r$ and $r'$ respectively, what is the radius of the circle of intersection of the two cones?

**Solution.** § 300, ⊿s $AOB$ and $CO'B$ are similar;

hence, § 299, $AB : BO' = r : r'$;

∴ § 276, $AB + BO' : BO' = r + r' : r'$. (1)

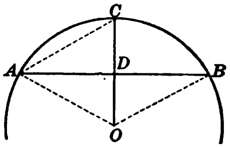


Since the cones are cones of revolution having the same axis $OO'$, any point in either surface is in the circumference of a ⊙ whose radius is perpendicular to $OO'$; hence, $BD$ ($r''$), the radius of the ⊙ of intersection, is perpendicular to $OO'$, and, § 302, ⊿s $AOO'$ and $BDO'$ are similar; consequently, $AO' : BO' = r : r''$; that is, $AB + BO' : BO' = r : r''$; (2)

hence, from (1) and (2), $r + r' : r' = r : r''$;

whence, $r'' = \dfrac{rr'}{r + r'}$.

Ex. **199**. A stone bridge 20 ft. wide has a circular arch of 140 ft. span at the water level. The crown of the arch is $140\,(1 - \frac{1}{2}\sqrt{3})$ ft. above the surface of the water. How many square feet of surface must be gone over in cleaning so much of the under side of the arch as is above water?



**Solution.** Let $AB = 140$ ft.,

and $CD = 140\,(1 - \frac{1}{2}\sqrt{3})$ ft. $= 70\,(2 - \sqrt{3})$.

Then, § 349,

$$AC = \sqrt{\overline{AD}^2 + \overline{CD}^2} = \sqrt{70^2 + 70^2(2 - \sqrt{3})^2}$$
$$= 70\sqrt{1 + (4 - 4\sqrt{3} + 3)} = 70\sqrt{4\,(2 - \sqrt{3})}$$
$$= 140\sqrt{2 - \sqrt{3}} \text{ ft.}$$

But, Ex. 712, $140\sqrt{2 - \sqrt{3}}$ is the side of a regular inscribed dodecagon whose radius is 140 ft.;

hence, $\angle AOC = \frac{1}{12}$ of $360° = 30°$, and $\angle AOB = 60°$;

∴ arc $ACB = \frac{1}{6}$ of the circum.;

but the circum. $= 2\,\pi R = 2 \times 3.1416 \times 140 = 879.648$ ft.;

hence, arc $ACB = \frac{1}{6}$ of 879.648 ft. $= 146.608$ ft.;

consequently, the surface to be gone over, being part of the surface of a cylinder, $= 146.608 \times 20 = 2932.16$ sq. ft.

Ex. **200**. What part of the whole surface of a sphere is a spherical triangle whose angles are 57° 57′, 75° 27′, and 100° 36′?

**Solution.** § 706, $E = (57° 57' + 75° 27' + 100° 36') - 180° = 54°$;

∴ § 737, area of $\triangle$ : area of surf. of sphere $= 54 : 720$;

hence, $\triangle$ is $\frac{54}{720}$, or $\frac{3}{40}$ of the surf. of the sphere.

Ex. **201**. What is the volume of a right cone whose altitude is 15 ft., inscribed in a sphere whose radius is 10 ft.?

**Solution.** Only a right circular cone can be inscribed in a sphere. Denote the radius of the sphere by $R$, that of the cone by $r$, and its altitude by $H$.

Then, the distance from the center of the sphere to the base of the cone

$$= H - R = 15 - 10 = 5 \text{ ft.},$$

and $r = \sqrt{R^2 - 5^2} = \sqrt{100 - 25} = \sqrt{75}$.

∴ § 630, vol. of cone $= \frac{1}{3}\,\pi r^2 H = \frac{1}{3} \times 3.1416 \times 75 \times 15 = 1178.1$ cu. ft.

Ex. **202**. How far from the base of a hemisphere must a plane be passed to divide the surface into two equivalent zones?

**Solution.** Since the area of each zone is $2\,\pi RH$, in order to be equivalent their altitudes must be equal; hence, the plane must be passed at a distance from the base equal to one half the radius of the sphere.

**Ex. 203.** The volume of a spherical segment of one base is $V$, and its altitude is $h$. What is the radius of the sphere?

**Solution.** § 750, $V = \frac{1}{2}\pi r^2 h + \frac{1}{6}\pi h^3$; whence, $r^2 = \dfrac{2(V - \frac{1}{6}\pi h^3)}{\pi h}$;

$$R^2 = r^2 + (R-h)^2 = \frac{2(V - \frac{1}{6}\pi h^3)}{\pi h} + R^2 - 2Rh + h^2;$$

whence, $$R = \frac{V - \frac{1}{6}\pi h^3}{\pi h^2} + \frac{h}{2} = \frac{3V + \pi h^3}{3\pi h^2}.$$

**Ex. 204.** Find an expression for the volume of a cube inscribed in a sphere whose radius is $r$.

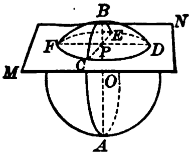
**Solution.** Denote the edge of the cube by $e$.

Then, Ex. 809, diag. of cube $= e\sqrt{3}$; but diag. of cube $= 2r$;

hence, $e\sqrt{3} = 2r$; whence, $e = \dfrac{2r}{\sqrt{3}}$,

and $$V = e^3 = \left(\frac{2r}{\sqrt{3}}\right)^3 = \frac{8r^3}{3\sqrt{3}}.$$

**Ex. 205.** Two equal circles intersect in a diameter. If a plane is passed perpendicular to that diameter, prove that the four points in which it intersects the circumferences lie in the circumference of a circle.



**Data:** Two equal Ⓢ intersecting in the diameter $AB$; also the plane $MN \perp AB$ at $P$ and intersecting the circumferences in $C$, $D$, $E$, $F$.

**To prove** $C$, $D$, $E$, $F$ in the circumference of a ⊙.

**Proof.** $MN$ intersects the plane of one ⊙ in the chord $FPD$, and that of the other in chord $CPE$; hence, since the Ⓢ have the common center $O$, and, § 443, $OP \perp FD$ and $CE$, § 202, $FD = CE$, and § 200, $OP$ bisects $FD$ and $CE$; consequently, Ax. 7, $PF = PC = PD = PE$; that is, § 173, $C$, $D$, $E$, $F$ are in the circumference of a ⊙ whose center is $P$.

**Ex. 206.** The square on the diameter of a sphere and the square on the edge of an inscribed cube are to each other as 3 is to 1.

**Proof.** Denote the diameter of the sphere by $D$ and the edge of the cube by $E$.

Then, since $D$ equals the diagonal of the cube,

Ex. 809, $D = E\sqrt{3}$; hence, $D^2 = 3E^2$;

consequently, § 273, $D^2 : E^2 = 3 : 1$.

**Ex. 207.** Find an expression for the altitude of a zone of a sphere whose radius is $r$, the area of the zone being equal to that of a great circle of the sphere.

**Solution.** § 398, area of gt. $\odot = \pi r^2$;
and, § 728, area of zone $= 2\pi rH$; hence, data, $2\pi rH = \pi r^2$;
whence, $H = \frac{r}{2}$.

**Ex. 208.** Find an expression for the altitude of a zone whose area is $A$ on a sphere whose volume is $V$.

**Solution.** § 744, $V = \frac{4}{3}\pi R^3$; whence, $R^3 = \frac{3V}{4\pi}$, and $R = \sqrt[3]{\frac{3V}{4\pi}}$;
hence, § 728, $A = 2\pi\sqrt[3]{\frac{3V}{4\pi}}H$; whence, $H = \frac{A}{2\pi\sqrt[3]{\frac{3V}{4\pi}}} = \frac{A}{\sqrt[3]{6V\pi^2}}$.

**Ex. 209.** Assuming the atmosphere to extend to a height of 50 miles above the earth's surface, and the earth to be a sphere whose radius is approximately 4000 miles, what is the volume of the atmosphere?

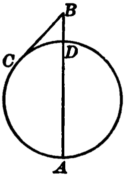
**Solution.** Denote the volume of the earth by $V$, and that of the earth plus that of the atmosphere by $V'$.

Then, § 744, $V = \frac{4}{3}\pi\, 4000^3$, and $V' = \frac{4}{3}\pi\, 4050^3$,
and volume of the atmosphere $= V' - V = \frac{4}{3}\pi\, 4050^3 - \frac{4}{3}\pi\, 4000^3$

$$= \tfrac{4}{3}\pi\,(4050^3 - 4000^3)$$
$$= \tfrac{4}{3} \times 3.1416 \times 2430125000$$
$$= 10179307600 \text{ cubic miles.}$$

**Ex. 210.** Assuming the earth to be a sphere whose radius is approximately 4000 miles, how far at sea is a lighthouse visible, if it is 80 ft. high?

**Solution.** Let $BC =$ the required distance, and $BD =$ 80 ft. $= \frac{1}{66}$ mi.; then, $AB = AD + BD = 8000\frac{1}{66}$ mi.
§ 315, $AB : BC = BC : BD$;
whence, $\overline{BC}^2 = AB \times BD$;
that is, $\overline{BC}^2 = 8000\frac{1}{66} \times \frac{1}{66} = \frac{528001}{66^2}$,
and $BC = \frac{\sqrt{528001}}{66} = 11.0096+$ mi.

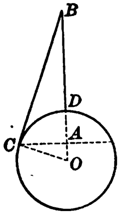


**Ex. 211.** A swimmer whose eye is at the surface of the water can just see the top of a buoy a mile distant. If the buoy is 8 in. out of the water, what is the radius of the earth?

**Solution.** (See Fig. for Ex. 210, Rev.) Let $BC = 1$ mi., and $BD =$ 8 in. $= \frac{1}{7920}$ mi.

§ 315, $AB : BC = BC : BD$; whence, $AB = \frac{\overline{BC}^2}{BD} = \frac{1}{\frac{1}{7920}} = 7920$ mi.;
consequently, $AD = 2R = 7920$ mi. $-$ 8 in.;
hence, $R = 3960$ mi. less 4 in.

Ex. **212.** How high above the surface of the earth must a man be in order that he may see $\frac{1}{n}$ of it?



**Solution.** Area of zone visible $= 2\pi RH$;

but, data, its area $= \frac{1}{n}$ of surface of sphere $= \frac{1}{n}4\pi R^2$;

hence, $2\pi RH = \frac{1}{n}4\pi R^2$; whence, $H = \frac{2R}{n}$,

and $OA = OD - AD = R - H = R - \frac{2R}{n} = \frac{R(n-2)}{n}$.

§ 313, $OB : OC = OC : OA$; whence, $OB = \frac{\overline{OC}^2}{OA} = \frac{R^2}{\frac{R(n-2)}{n}} = \frac{Rn}{n-2}$.

Then, $BD = OB - OD = \frac{Rn}{n-2} - R = \frac{2R}{n-2}$.

Ex. **213.** What is the area of the zone illuminated by a taper $h$ decimeters from the surface of a sphere whose radius is $r$ decimeters?

**Solution.** (See Fig. for Ex. 212, Rev.)

§ 313, $OB : OC = OC : OA$; whence, $OA = \frac{\overline{OC}^2}{OB} = \frac{r^2}{r+h}$;

then, $H = AD = OD - OA = r - \frac{r^2}{r+h} = \frac{rh}{r+h}$.

§ 728, area of zone $= 2\pi RH = 2\pi r\frac{rh}{r+h} = \frac{2\pi r^2 h}{r+h}$ sq. dm.

Ex. **214.** In a cube whose edge is 1 ft. there are inscribed a cylinder, a cone, a sphere, and a square pyramid. What is the volume of each of these solids?

**Solution.**

(1) § 606, vol. of cylinder $= \pi R^2 H = 3.1416 \times (\frac{1}{2})^2 \times 1 = .7854$ cu. ft.

(2) § 630, vol. of cone $= \frac{1}{3}\pi R^2 H = \frac{1}{3} \times 3.1416 \times (\frac{1}{2})^2 \times 1 = .2618$ cu. ft.

(3) § 744, vol. of sphere $= \frac{4}{3}\pi R^3 = \frac{4}{3} \times 3.1416 \times (\frac{1}{2})^3 = .5236$ cu. ft.

(4) § 560, vol. of pyramid $= \frac{1}{3}B \times H = \frac{1}{3} \times 1^2 \times 1 = \frac{1}{3}$ cu. ft.

Ex. **215.** A cylindrical boiler with hemispherical ends has a total length of 12 ft., and its circumference is 10 ft. What is its surface and the weight of the water required to fill it?

**Solution.** $C = 2\pi R = 10$; whence, $R = \frac{5}{\pi}$,

and the surface of the ends is that of two equal hemispheres, or that of a sphere whose radius is $\frac{5}{\pi}$;

hence, surface of both ends $= 4\pi R^2 = 4\pi \times \left(\frac{5}{\pi}\right)^2 = \frac{100}{\pi}$ sq. ft.

$$\text{Length of cylindrical part} = 12 - 2R = 12 - \frac{10}{\pi};$$

then,

surface of cylindrical part $= 2\pi RH = 2\pi\frac{5}{\pi}\left(12 - \frac{10}{\pi}\right) = \left(120 - \frac{100}{\pi}\right)$ sq. ft.

Hence, entire surface of boiler $= \frac{100}{\pi} + 120 - \frac{100}{\pi} = 120$ sq. ft.

$$\text{Vol. of hemispheres} = \tfrac{4}{3}\pi R^3 = \tfrac{4}{3}\pi\left(\frac{5}{\pi}\right)^3 = \frac{500}{3\pi^2} \text{ cu. ft.}$$

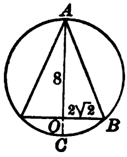
$$\text{Vol. of cylindrical part} = \pi R^2 H = \pi\left(\frac{5}{\pi}\right)^2\left(12 - \frac{10}{\pi}\right) = \left(\frac{300}{\pi} - \frac{250}{\pi^2}\right) \text{ cu. ft.}$$

Hence, total vol. of boiler $= \frac{500}{3\pi^2} + \frac{300}{\pi} - \frac{250}{\pi^2} = \frac{300}{\pi} - \frac{250}{3\pi^2}$

$= 95.49 - 25.33 = 70.16+$ cu. ft.

Hence, the weight of water required to fill it is $62\frac{1}{2}$ lb. × 70.16+ = 4385+ lb.

Ex. **216.** Find the diameter of a sphere which is circumscribed about a regular square pyramid whose base is 4 in. square and altitude 8 in.



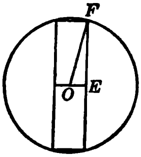
**Solution.** § 351, diag. of base $= 4\sqrt{2}$; this is also a chord of the great ⊙ of the sphere made by a plane through two diagonally opposite edges; then, $OB = \frac{1}{2} \times 4\sqrt{2} = 2\sqrt{2}$.

§ 312, $AO : OB = OB : OC$;

that is, $8 : 2\sqrt{2} = 2\sqrt{2} : OC$;

whence, $OC = \frac{2\sqrt{2} \times 2\sqrt{2}}{8} = 1$, and diam. $AC = AO + OC = 8 + 1 = 9$ in.

Ex. **217.** A sphere 8 in. in diameter has a 3-inch hole bored through its center. What is the remaining volume?



**Solution.** The part bored out consists of a cylinder and two equal spherical segments one at each end of the cylinder. Denote the radius of the sphere by $R = 4$ in., and of the cylinder by $r = \frac{3}{2}$ in., and the altitude of each segment by $h$.

§ 350, $EF = \sqrt{\overline{OF}^2 - \overline{OE}^2} = \sqrt{4^2 - (\frac{3}{2})^2} = \sqrt{16 - \frac{9}{4}} = \frac{1}{2}\sqrt{55}$;

hence, the alt. of the cylinder $= 2 \times \frac{1}{2}\sqrt{55} = \sqrt{55} = 7.416+$ in.,

and, § 606, vol. of cylinder $= \pi r^2 H = 3.1416 \times (\frac{3}{2})^2 \times 7.416+ = 52.4207376$ cu. in.

Altitude $h$ of each segment $= \frac{1}{2}$ (8 in. − 7.416 in.) = .292 in.

Now, § 750, vol. of both segments $= 2\left(\frac{1}{2}\pi r^2h + \frac{1}{6}\pi h^3\right) = \frac{1}{3}\pi h\left(3r^2 + h^2\right)$
$= \frac{1}{3} \times 3.1416 \times .292 \times \left[3\left(\frac{3}{2}\right)^2 + .292^2\right]$
$= \frac{1}{3} \times 3.1416 \times .292 \times 6.835264$
$= 2.0901034+$ cu. in.

**Then,** total vol. bored out $= 52.4207376$ cu. in. $+ 2.0901034+$ cu. in.
$= 54.51084+$ cu. in.

Vol. of sphere $= \frac{4}{3}\pi R^3 = \frac{4}{3} \times 3.1416 \times 4^3 = 268.0832$ cu. in.

**Hence,** the remaining vol. $= 268.0832$ cu. in. $- 54.51084$ cu. in.
$= 213.57236$ cu. in. nearly.

**Ex. 218.** What is the volume of the portion of a sphere lying outside of an inscribed cylinder of revolution whose altitude is $h$ and radius $r$?

**Solution.** Denote the radius of the sphere by $R$.

Then,
$$R = \sqrt{r^2 + \left(\frac{h}{2}\right)^2} = \frac{1}{2}\sqrt{4r^2 + h^2};$$

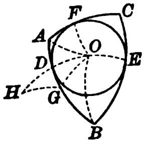
hence, vol. of sphere $= \frac{4}{3}\pi R^3 = \frac{4}{3}\pi\left(\frac{1}{2}\sqrt{4r^2 + h^2}\right)^3 = \frac{\pi}{6}\left(\sqrt{4r^2 + h^2}\right)^3.$

Vol. of cylinder $= \pi r^2h.$

Hence, required vol. $= \frac{\pi}{6}\left(\sqrt{4r^2 + h^2}\right)^3 - \pi r^2h = \frac{\pi}{6}\left[\left(\sqrt{4r^2 + h^2}\right)^3 - 6r^2h\right].$

**Ex. 219.** Inscribe a circle in a given spherical triangle.

**Solution.** Ex. 947, bisect ∡ $A$ and $B$ of the given spherical △ $ABC$ and produce the bisectors to intersect as at $O$; Ex. 954, draw the great ⊙ arc $OD \perp AB$.



With $O$ as pole and $OD$ as arc radius, describe the ⊙ $DEF$.

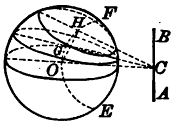
Then, $DEF$ is the ⊙ required.

**Proof.** On arc $BC$ take $BE = BD$, and on arc $AC$ take $AF = AD$; draw the great ⊙ arcs $OE$ and $OF$; then, since two sides and the included ∠ of △ $OBE$ equal two sides and the included ∠ of △ $OBD$, △ $OBE$ can be made to coincide with the symmetrical of △ $OBD$; consequently, ⩓ $OBE$ and $OBD$ are symmetrical; ∴ § 708, $OE = OD$, and $\angle OEB = \angle ODB$; that is, $OE \perp BC$; similarly, $OF = OD$, and $OF \perp AC$.

Draw any other great ⊙ arc from $O$ to $AB$, as $OG$; produce $OD$ to $H$, making $DH = OD$, and draw the great ⊙ arc $HG$; then, as above, ⩓ $HDG$ and $ODG$ are symmetrical, and $HG = OG$.

Now, § 696, $OG + HG > OH$; that is, $2\,OD < 2\,OG$, or $OD < OG$; similarly, $OE$ and $OF$ are less than any other great ⊙ arcs from $O$ to $BC$ and $AC$ respectively; hence, the ⊙ $DEF$ lies within the △ $ABC$ and touches its sides at $D$, $E$, and $F$.

Ex. **220.** Find the locus of the centers of the sections of a given sphere made by planes passing through a given straight line.



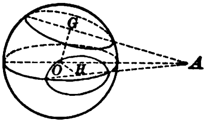
**Solution.** Let *O* be the center of the sphere and of the great ⊙ section cut by the given plane, and let *G* and *H* be the centers of the Ⓢ which are sections made by any two of the cutting planes.

Since, § 644, *OG* ⊥ plane of the ⊙ whose center is *G*, and *OH* ⊥ plane of the ⊙ whose center is *H*, § 482, the plane determined by *OG* and *OH* is perpendicular to the plane of these Ⓢ, and therefore, § 485, perpendicular to *AB*, the given line, intersecting it as at *C*; that is, the centers of the sections lie in a plane through *O* and ⊥ *AB*.

Draw *CO*, *CG*, and *CH*.

Since, § 443, *OG* ⊥ *CG* and *OH* ⊥ *CH*, *G* and *H* are the vertices of rt. △ on the base *OC* as a hypotenuse; that is, Ex. 388, the centers of the sections lie on a circumference whose diameter is *OC* and the required locus is the arc *EOGF* of that circumference lying within the given sphere.

Ex. **221.** Find the locus of the centers of the sections of a given sphere made by planes passing through a given point without the sphere.



**Solution.** Let *O* be the center of the given sphere, and *G* and *H* the centers of Ⓢ which are sections made by any two of the cutting planes.

Since *OG* ⊥ plane of the ⊙ whose center is *G*, and *OH* ⊥ plane of the ⊙ whose center is *H*, § 443, *OG* and *OH* are perpendicular respectively to the lines *AG* and *AH* drawn in the planes of the Ⓢ from the given point, and the centers of the sections are the feet of ⊥s from *O* to lines in space that meet at *A*; consequently, Ex. 945, these centers lie in the surface of a sphere, and since *OA* is the common base and hypotenuse of the rt. △ *OGA*, *OHA*, etc., *OA* is the diameter of this sphere.

Hence, the required locus is the zone of the sphere whose diameter is *OA*, lying within the given sphere.

Ex. **222.** Having given the radius, construct a spherical surface to pass through three given points.

**Solution.** Through the given points pass the circumference of a ⊙, and at its center erect a ⊥ to the plane of the ⊙; find a point in this ⊥ at the distance of the given radius from one of the given points.

Then, § 450, this point is at the same distance from the other given points, and it is the center of a spherical surface of given radius passing through them.

**Sch.** Since another point equidistant from the given points may be found on the other side of their plane, there are two solutions of this problem.

Ex. **223**. Having given the radius, construct a spherical surface to pass through two given points and be tangent to a given plane or to a given sphere.

**Solution.** (1) With the given points $A$ and $B$ as centers, construct spheres with radii = the given radius $r$; construct planes ∥ $MN$, the given plane, at the distance $r$ from $MN$.

With a point common to the surfaces of these spheres and either of these planes, as center, construct a sphere of radius $r$.

Then, the surface of this sphere is the spherical surface required.

**Proof.** Since, const., $A$ and $B$ are at the distance $r$ from the center of this sphere, its surface passes through $A$ and $B$, and since its center is at the distance $r$ from $MN$ it is tangent to $MN$.

**Sch.** If the intersection of the surfaces of the spheres whose centers are $A$ and $B$ lies in one of the planes ∥ $MN$, the problem cannot be determined.

If the intersection of these spherical surfaces is cut by one of the planes ∥ $MN$, there are two solutions; if it touches one of those planes, there is only one solution.

If the spheres whose centers are $A$ and $B$ have no point in common with either of the planes ∥ $MN$, there is no solution.

If $A$ and $B$ are on opposite sides of $MN$, or if both are in $MN$, there is no solution.

(2) Let $O$ be the center of the given sphere whose radius is $R$.

With $O$ as common center, construct spheres with the sum and difference of $r$ and $R$ as radii respectively.

With a point common to the surfaces of the spheres whose centers are $A$ and $B$, and either of the spheres concentric with the given sphere, as center, construct a sphere of radius $r$.

Then, the surface of this sphere is the spherical surface required.

**Proof.** As in (1), this spherical surface passes through $A$ and $B$, and since, const., its center is at the distance $r$ from the given sphere, it is tangent to that sphere.

**Sch.** Since the spherical surfaces whose centers are $A$ and $B$ may have four, three, two, one, or no points in common with the constructed concentric spherical surfaces, there may be four, three, two, one, or no solutions accordingly.

Ex. **224**. Having given the radius, construct a spherical surface to pass through a given point and be tangent to two given planes.

**Solution.** With the given point $A$ as center, construct a sphere with radius = the given radius $r$.

On the same side of the given plane $MN$, as $A$, construct a plane ∥ $MN$ and at the distance $r$ from $MN$; on the same side of the other given plane $PQ$, as $A$, construct a plane ∥ $PQ$ and at the distance $r$ from $PQ$.

With a point common to the surface of the sphere whose center is *A* and the intersection of the constructed planes as center, construct a sphere whose radius is *r*.

Then, the surface of this sphere is the spherical surface required.

**Proof.** Since, const., *A* is at the distance *r* from the center of this sphere, its surface passes through *A*; and since, const., the perpendicular distance from its center to each of the planes *MN* and *PQ* is *r*, this sphere is tangent to *MN* and *PQ*.

**Sch.** If *MN* and *PQ* intersect, and *A* lies within either of the dihedral $\measuredangle$ formed by them, there may be two, one, or no points in common with the intersection of the constructed planes and the surface of the sphere whose center is *A*; consequently, there may be two, one, or no solutions accordingly; but if *A* is in either given plane *MN* or *PQ* there is no solution unless the $\perp$ to the given plane at *A* cuts either intersection of the constructed planes at the distance *r* from *A*.

If *MN* and *PQ* are parallel at the distance 2 *r* from each other and *A* lies between them, there will be an indefinite number of indeterminate solutions; but if *A* lies in either *MN* or *PQ* there is only one solution. If *A* does not lie between the planes, or in one of them, or if the distance between *MN* and *PQ* is not 2 *r*, there is no solution.

Ex. **225.** Having given the radius, construct a spherical surface to be tangent to three given planes.

**Solution.** Construct two planes parallel to one given plane *MN* and at a distance from it equal to the given radius *r*.

Also construct planes parallel to the other given planes *PQ* and *RS* at the distance *r* from each.

With a point, common to any three of the six constructed planes, as center, construct a sphere of radius *r*.

Then, the surface of this sphere is the spherical surface required.

**Proof.** Since, const., the perpendicular distance from the center of this sphere to each of the planes *MN*, *PQ*, and *RS* is *r*, the sphere is tangent to each of these planes.

**Sch.** If *MN*, *PQ*, and *RS* intersect in a single point, there are eight different solutions.

If two of the given planes are parallel and at a distance apart equal to 2 *r*, there may be an unlimited number of solutions, the centers of the spheres being in lines, one on each side of the third given plane, midway between the other two and parallel to the intersections of the given planes.

If the given planes are all parallel, or have a line in common, there is no solution.

## ANSWERS TO QUESTIONS IN PROOFS

**The answers are given under each section in the order of the questions.**

§ 103 . . . . . . . § 103, 1.

§ 115 *a* . . . . . . . . . § 76.
*b* . . . . . . . . . § 73.

§ 131 . . . . . . . . . § 124.

§ 158 . . . . . . . . . § 151.

§ 170. Any two of the ⊥s intersect, otherwise they would be parallel and the sides to which they are perpendicular would be in the same straight line, being ⊥s to two parallel lines from the same point (a vertex of the $\triangle$); but this is impossible, since they are sides of a $\triangle$.

§ 196 . . . . . . . . . § 107.

§ 198 *a* . . . . . . . Ax. 14.
*b* . . . . . . . . . § 108.

§ 202 *a* . . . . . . . . . § 108.
*b* . . . . . . . . . § 108.

§ 203 *a*. Since, § 110, the sum of the $\angle$s of $\triangle ABO = 2$ rt. $\angle$s, and the sum of the $\angle$s of $\triangle DEP = 2$ rt. $\angle$s, the sum of the $\angle$s of $\triangle ABO =$ the sum of the $\angle$s of $\triangle DEP$; $\therefore$ taking $\angle AOB$ from the sum of the $\angle$s of $\triangle ABO$, and $\angle DPE$ from the sum of the $\angle$s of $\triangle DEP$, $\angle A + \angle B$ is less than $\angle D + \angle E$;

but, § 116, $\angle A = \angle B$, and $\angle D = \angle E$;

then, $2\angle A$ is less than $2\angle D$;

that is, $\angle A$ is less than $\angle D$,

or $\angle PDE$ is greater than $\angle OAB$.

*b* . . . . . . . Ax. 14.
*c* . . . . . . . . . § 129.
*d* . . . . . . . . . § 61.

§ 205 . . . . . . . . . § 61.

§ 209 *a* . . . . . . . Ax. 14.
*b* . . . . . . . . . § 123.

§ 213. § 26, $\angle OCA$ is a rt. $\angle$, and $\angle PCA$ is a rt. $\angle$;

$\therefore \angle OCA + \angle PCA = 2$ rt. $\angle$s.

§ 225 *a* . . . . . . . Ax. 14.
*b* . . . . . . . . . § 116.

§ 238 . . . . . . . . . § 108.

§ 245 . . . . . . . . . § 110.

§ 247 . . . . . . . . . § 124.

Ex. 369 . . . . . . . . . § 134.

§ 292 *a* . . . . . . . . . § 76.
*b* . . . . . . . . . § 73.
*c* . . . . . . . . . Ax. 1.

§ 293 *a* . . . . . . . . . § 73.
*b* . . . . . . . . . § 76.
*c* . . . . . . . . . Ax. 1.

§ 294. If the $\triangle$ were isosceles or equilateral, $CE$ would be parallel to $AB$ and consequently would not intersect $AB$ produced.

§ 298 . . . . . . . . . Ax. 1.

§ 300 *a* . . . . . . § 108, data.
*b* . . . . . . . . . § 77.

§ 303 . . . . . . . . . § 76.

§ 305 . . . . . . . . . Ax. 1.

§ 306 *a*. Since, const., $GC = DF$, and $HC = EF$.
*b* . . . . . . . . . § 76.

§ 308 *a* . . . . . . . . . § 76.
*b*. By first part of proof.

§ 309 *a* . . . . . . . . . § 299.
*b* . . . . . . . . . Ax. 2.

§ 310 *a* . . . . . . . . . § 299.
*b* . . . . . . . . . § 299.
*c* . . . . . . . . . § 299.
*d* . . . . . . . . . Ax. 3.

§ 310 *e* . . . . . . . . . § 299.
*f* . . . . . . . . . § 299.
*g* . . . . . . . . . Ax. 1.

§ 311 . . . . . . . . § 299.

§ 312. Triangles which are similar to the same triangle must be similar to each other.

§ 315. Since they have the same measure.

§ 327 . . . . . . . . . Ax. 1.

§ 331 *a* . . . . . . . . . § 153.
*b* . . . . . . . . . § 153.
*c* . . . . . . . . . § 123.

§ 347 . . . . . . . . . Ax. 1.

§ 348 *a*. Being an $\angle$ of a square.

*b*. Since, § 55, $\angle ABG + \angle GBC = 2$ rt. $\angle$s and $\angle GBC =$ a rt. $\angle$, Ax. 3, $\angle ABG =$ a rt. $\angle$.

*c* . . . . . . . . . § 58.

*d*. Being $\angle$s of a square.

*e*. Since their sides are parallel to the sides of $\angle$s *AFH* and *BHF* respectively, § 81, they are equal respectively to $\angle$s *AFH* and *BHF*, which are $\angle$s of a square.

*f* . . . . . . . . . § 143.
*g* . . . . . . . . . § 143.

§ 349 *a* . . . . . . . . . § 143.
*b* . . . . . . . . . Ax. 2.

§ 357 . . . . . . . . . § 299.

§ 359 *a*. Since, § 225, each is measured by ½ arc *ED*.

*b* . . . . . . . . . § 301.
*c* . . . . . . . . . § 299.

§ 360 *a*. Since, § 225, each is measured by ½ arc *BC*.

*b* . . . . . . . . . § 301.

§ 361. Since, § 225, each is measured by ½ arc *BC*.

§ 365 . . . . . . . . . § 349.

§ 366 . . . . . . . . . § 269.

§ 367 *a*. Since, § 151, their altitudes are equal, § 336, the $\triangle$s are equivalent.

*b*. Since, § 151, their altitudes are equal, § 336, the $\triangle$s are equivalent.

§ 370. § 71, $DB \parallel EF$;

∴ § 151, $DB = EF$.

§ 372 . . . . . . . . . § 288.

§ 375. Since, § 225, their measures are equal.

§ 376 *a* . . . . . . . . . § 196.

*b*. Since, § 225, their measures are equal.

*c*. Since, § 231, they are measured by halves of the equal arcs *AB*, *BC*, etc.

*d* . . . . . . §§ 102, 118.
*e* . . . . . . §§ 108, 90.

§ 381 *a* . . . . . . . . . Ax. 14.
*b* . . . . . . . . . § 116.

§ 384 *a*. Since, § 374, chord $AB =$ chord $BC =$ etc., § 196, arc $AB =$ arc $BC =$ etc.

*b* . . . . . . Data, Ax. 7.

§ 387 *a*. Since $OA = OB$, and $PF = PG$, $OA : OB = PF : PG$, that is, § 274, $OA : PF = OB : PG$.

*b* . . . . . . . . . § 305.

§ 392 *a* . . . . . . . . . § 277.
*b* . . . . . . § 349, Ax. 3.

§ 402 *a* . . . . . . . . . § 396.

*b*. Since $AO = CO$, and $DP = FP$.

*c* . . . . . . . . . § 306.
*d* . . . . . . . . . § 274.
*e* . . . . . . . . . § 274.
*f* . . . . . . . . . Ax. 1.

§ 405 *a* . . . . . . . . . § 306.
*b* . . . . . . § 90, const.
*c* . . . . . . . . . § 115.

§ 411 *a* . . . . . . . . . § 335.
*b* . . . . . . § 61, data.

§ 411 *c*. Since $AC > ED$ and the other factors of the two products are equal each to each.

§ 412 *a*, § 90,
$BC = AC$, const., $CE = AC$;
∴ $BC = CE$; then, $AE = AC + BC$;
data, $AC + BC = AD + BD$;
and, const., $BD = FD$;
consequently, $AE = AC + BC = AD + BD = AD + FD$.

| | | |
|---|---|---|
| | *b* | § 125. |
| | *c* | § 122. |
| | *d* | § 335. |
| § 424 | *a* | § 107. |
| | *b* | § 108. |
| § 425 | *a* | § 151. |
| | *b* | § 70. |
| § 447 | *a* | § 443. |
| | *b* | § 447, 1. |
| § 466 | | § 72. |
| § 469 | *a* | §§ 153, 140. |
| | *b* | §§ 140, 153. |
| | *c* | § 107. |
| § 488 | | § 114. |
| § 489 | | Const., § 61. |
| § 490 | | § 72. |
| § 498 | *a* | §§ 100, 108. |
| | *b* | § 125. |
| | *c* | § 130. |

§ 499. The sum of all the ∡ of the △ whose vertex is $Q =$ the sum of all the ∡ of the △ whose vertex is $O$; consequently, taking a greater sum from the first side of the equality than from the second, there will remain a less sum on the first side of the resulting inequality than on the second; that is, the sum of the face ∡ at $Q$ is less than the sum of the ∡ at $O$.

| | | |
|---|---|---|
| § 500 | *a* | § 100. |
| | *b* | §§ 108, 107. |
| | *c* | § 108. |
| | *d* | § 114. |
| § 527 | | § 76. |
| § 528 | *a* | § 140. |
| | *b* | § 153. |
| | *c* | § 155. |
| § 529 | | § 150. |
| § 538 | | §§ 151, 333. |
| § 540 | | § 152. |
| § 551 | | § 334. |
| § 555 | | § 274. |
| § 558 | | § 152. |
| § 562 | *a* | § 561. |
| | *b* | § 102. |
| | *c* | § 108. |
| | *d* | § 299. |
| § 565 | *a* | §§ 333, 457. |
| | *b* | § 561. |
| § 568 | | § 302. |
| § 573 | | § 303. |
| § 597 | *a* | § 153. |
| | *b* | § 107. |
| § 621 | | Ax. 14. |

§ 689. § 224, $\angle AOP = 90°$; this is, $OA \perp PP'$.

| | | |
|---|---|---|
| § 712 | | § 708. |
| § 713 | | § 686, const. |

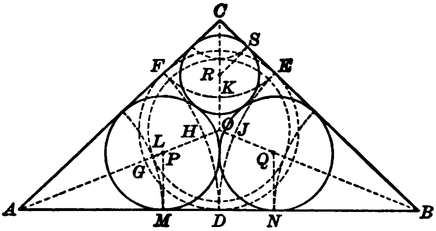
# EXERCISES WHICH APPEAR ONLY IN THE FIRST EDITION OF THE GEOMETRY

**Ex. 183. Can two circles have a common center without coinciding?**

**Yes, for concentric Ⓢ having different radii do not coincide though, § 174, they have the same center.**

**Ex. 735. In an isosceles triangle describe three circles touching each other, and each touching two of the three sides of the triangle.**

**Solution.** In the given isos. $\triangle ABC$ inscribe a ⊙; denote the points of contact of the sides $AB$, $BC$, and $AC$, by $D$, $E$, and $F$, respectively, and the center of the ⊙ by $O$.



Draw $OA$, $OB$, and $OC$, $OA$ intersecting the circumference of the inscribed ⊙ as at $G$.

From $A$, $B$, and $C$, as centers, with radii $AD$, $BD$, and $CE$, respectively, describe the arcs $DF$, $DE$, and $EF$, intersecting $OA$, $OB$, and $OC$, as in $H$, $J$, and $K$, respectively.

On $OA$ take $OL = OH + OJ + OK$; find the middle point of $GL$, and through it pass the circumference of a ⊙ having its center at $O$; with $A$, $B$, and $C$, as centers, describe arcs tangent to this ⊙, the first and second meeting $AB$ as at $M$ and $N$, respectively, and the third meeting $BC$ as at $S$.

At $M$ and $N$ erect perpendiculars to $AB$ meeting $OA$ and $OB$ as at $P$ and $Q$, respectively; also erect a perpendicular to $BC$ at $S$ meeting $OC$ as at $R$.

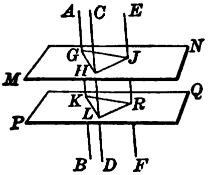
With $P$, $Q$, and $R$, as centers, and radii $PM$, $QN$, and $RS$, respectively, describe Ⓢ.

Then, these Ⓢ are the Ⓢ required.

[The proof is too difficult for elementary geometry, and for that reason the problem has been omitted.]

**Ex. 147. (Review.) If three parallel straight lines, not in the same**

**plane, pierce parallel planes, and the points in which each plane is pierced are joined, the triangles thus formed are all equal.**



**Data:** Parallel lines $AB$, $CD$, and $EF$ piercing parallel planes $MN$ and $PQ$ in $G$, $H$, $J$, and $K$, $L$, $R$, respectively, and the lines joining the points in each plane.

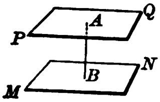
**To prove** $\triangle GHJ = \triangle KLR$.

**Proof.** Data, $GK \parallel HL$, and, § 464, $GK = HL$; hence, § 150, $GKLH$ is a $\square$, and, § 153, $GH = KL$.

Similarly, $HJ = LR$, and $GJ = KR$;

hence, § 107, $\triangle GHJ = \triangle KLR$.

**Ex. 148.** (Review.) **Through a given point pass a plane parallel to a given plane.**



**Solution.** From the given pt. $A$ draw $AB \perp MN$, the given plane, and through $A$ pass the plane $PQ \perp AB$.

Then, § 462, $PQ \parallel MN$.

**Ex. 182.** (Review.) **If one edge of a cube is $m$, what is the altitude of an equivalent cylinder of revolution whose diameter is $n$?**

**Solution.** Vol. of cube = vol. of cylinder = $m^3$;

but, § 630, vol. of cylinder $= \frac{1}{3}\pi\left(\frac{n}{2}\right)^2 H$; hence, $\frac{1}{12}\pi n^2 H = m^3$.

Therefore, $H = \frac{12\,m^3}{\pi n^2}$.

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# ADVERTISEMENTS